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Mobile Data Offloading with Flexible Pricing

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***Abstract*—We consider a data offloading scenario where the small-cell service providers (SSPs) are allowed to implement a *flexible-pricing scheme*. In the aforementioned scheme, the price that an SSP charges the mobile-network operator (MNO) depends on the amount of MNO's traffic that is offloaded onto the respective SSP. Formulating the SSPs' problem of determining their traffic-dependent prices as a Bayesian game, we first show that there exists no Nash equilibriums in pure strategies. We then proceed to derive the structure of a mixed-strategy symmetric Bayesian Nash equilibrium (BNE). We also compare the flexible-pricing scheme with the traditional *flat-pricing scheme* (where the SSPs are restricted to announce a single price, irrespective of the traffic that is offloaded onto them) in terms of the payoffs achieved by the SSPs as well as the MNO. We show that the SSPs benefit under the flexible-pricing scheme while the MNO incurs a loss in payoff; however, the net-payoff of the system remains balanced. Formally, in terms of mechanism design, the flexible-pricing scheme is *incentive compatible* as all entities (including the neutral MNO) achieve a non-negative payoff. Finally, focussing on the SSPs' payoff, we conduct a numerical study to demonstrate the efficacy of the flexible-pricing scheme over the flat-pricing scheme (using *price of anarchy* as the metric).**

I. INTRODUCTION

We have been witnessing an ever-increasing demand for mobile data. Indeed, according to the latest report by Cisco [1], global mobile-data traffic is expected to reach figures of about 77.5 exabytes per month by 2022 (1 exabyte = 10^9 gigabyte). In order to meet this huge demand for mobile data, one promising approach that researchers have been envisioning is that of *mobile data offloading* [2]–[4]. Mobile data offloading is a proposal whereby the mobile network operators (MNOs) are allowed to offload some of their users onto small-cell service providers (SSPs) e.g., femto-cell operators, public WiFi providers, etc. The MNOs can thus benefit by implicitly serving more traffic (equivalently, by providing subscription to more users), while the SSPs can generate revenue by charging the MNOs for providing offloading services. However, when multiple SSPs are present, the data offloading scenario leads to interesting *pricing problems* whereby the SSPs are expected to set competitive prices in order to make profit by serving sufficient offloaded data (see [5]–[8] and references therein).

In this work we propose a novel *flexible-pricing scheme* where the SSPs are allowed to charge the MNO differently for the different amount of traffic demands generated by the offloaded users. Alternatively, we assume that the demand for traffic flow is random, while the SSPs are allowed to set different prices for the different traffic flow it may actually experience. Our model is thus in contrast to the existing work in the literature where the flows are generally assumed deterministic (or simply use average flow into the formulation); moreover, *flat-pricing schemes* are implemented in the existing work where the SSPs are restricted to announce a single price, irrespective of the amount of traffic offloaded onto them [5].

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Our main technical contributions are as follows: (1) We model the above data offloading scenario (comprising random flows and involving flexible pricing) as a Bayesian game; the solution is characterized in terms of Bayesian Nash equilibrium (BNE). (2) We prove the non-existence of a BNE in pure-strategies (Theorem 1), and then proceed to derive the structure of a mixed-strategy symmetric BNE (Theorem 3). (3) We show that the SSPs achieve a higher payoff in the flexible-pricing scheme when compared with their payoff in the traditional flat-pricing scheme (Lemma 2); the MNO in contrast achieves a lower payoff (Lemma 3). However, the net-payoff of the system remains conserved. (4) We demonstrate the efficacy of the flexible pricing scheme (from the SSPs point-of-view) via. a numerical study.

The remainder of the paper is outlined as follows. In Section II we first present the details of the considered flexible-pricing data-offloading model, and then characterize the solution in terms of BNEs. The case of pure-strategy BNEs is study in Section III, while in Section IV we consider mixed-strategy BNEs. In Section V we compare the proposed flexible-pricing scheme with the existing flat-pricing scheme. Results from our numerical study are reported in Section VI. We finally present our conclusions in Section VII.

II. SYSTEM MODEL

We consider a system comprising one *Mobile Network Operator (MNO)* and two *Small-cell Service Providers (SSPs)*. The MNO can achieve data-offloading by handing-off some of its users to the SSPs for service. Specifically, the users that are within the coverage region of (the access points installed by) SSP-i ($i \in \{1, 2\}$) but not in the coverage region of SSP-j ($j \neq i$) are offloaded to SSP-i, whereas the users that lie in the overlap region can be offloaded to either of the SSPs. Let F_i ($i \in \{1, 2\}$) denote the aggregate traffic demand (henceforth referred to as *flow*) generated by all the users that are offloaded to SSP-i, while f_o is used to denote the flow generated by the users that lie in the overlap region.

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A. Flow Model

Our model has thus far been similar to the one considered in the recent work of Li et al. [5]. We however now deviate from their model by introducing randomness into the flows experienced at the SSPs. Further, we take into consideration the lack of information at SSP-i regarding the flow experienced at SSP-j. The formal details are as follows.

We assume that the values of the flows, F_1 and F_2, being private information, are known only to the respective SSPs; whereas, the overlap-flow value f_o is commonly known to both the SSPs. We model the above scenario by assuming that F_1 and F_2 are random variables, whose realized values are known only to the respective SSPs. For simplicity, we begin with a simple model where F_1 and F_2 are i.i.d. taking two values f_ℓ and f_h with probabilities θ and $(1-\theta)$, respectively. We assume that $f_\ell < f_h$, so that the probability that the flow F_i ($i=\{1,2\}$) is *low* (i.e., f_ℓ) is θ, while the flow is *high* (i.e., f_h) with probability $(1-\theta)$. The average flow is thus given by $f_a = \theta f_\ell + (1-\theta) f_h$. Finally, the overlap flow f_o is considered constant whose value is known to both SSPs.

Remark: For $\theta = 0$ or $\theta = 1$ our model degenerates to the model studied by Li et al. [5] where the flows are deterministic. Alternatively, the model in [5] can be thought as simply taking the average flow f_a into account while modeling the SSP flows, instead of considering the detailed distribution like in our model.

B. Pricing Strategy

The MNO charges its users at a fixed flat-price of p_M per-unit-flow of request for data. The SSPs on the other hand, are allowed to implement a *flexible-pricing scheme* whereby a SSP can charge different prices for the different flow levels (low or high) it would actually experience. Formally, let $p_i = (p_{i,\ell}, p_{i,h})$ denote the *price vector* of SSP-i ($i \in \{1,2\}$) where $p_{i,\ell}$ (resp. $p_{i,h}$) is the price per-unit-flow that SSP-i charges the MNO when the flow $F_i = f_\ell$ (resp. f_h). Thus, given a price vector p_i, the price set by SSP-i depends on the realized value of F_i. For convenience, we define the *price random variables* P_i as follows:

$$P_i \quad = \quad \begin{cases} p_{i,\ell} \text{ if } F_i = f_\ell \\ p_{i,h} \text{ if } F_i = f_h \end{cases}$$

Note that P_1 and P_2 are independent (since F_1 and F_2 are i.i.d.) and their respective p.m.fs are given by $P_i = p_{i,\ell}$ w.p. θ, and $P_i = p_{i,h}$ w.p. $(1-\theta)$. Finally, we use c_S to denote the cost incurred by a SSP to serve one-unit of flow.

Remark: The above flexible-pricing scheme is in contrast to the *flat-pricing scheme* implemented in [5] where the SSPs are restricted to set a single price, irrespective of the amount of flow offloaded onto them. The flat-pricing scheme is discussed in detail in Section V.

Now, since F_i can be served only by SSP-i, the SSPs have monopoly over their respective flows (assuming that the SSPs' prices are less than the MNO's price of p_M, otherwise the MNO has no incentive to offload). However, since the MNO would naturally offload the overlap flow f_o to the SSP that charges the lowest price[1], the SSPs are hence expected to set competitive prices so as to optimize their profits by additionally serving the overlap flow f_o.

C. Payoff Functions

Given the price vectors of both SSPs, the (random) payoff achieved by the MNO can be written as[2]

$$U_M(p_i, p_j) = F_i\Big(p_M - P_i\Big) + F_j\Big(p_M - P_j\Big) + f_o\Big(p_M - \min\{P_i, P_j\}\Big). \tag{1}$$

The average payoff is given by

$$\mathcal{U}_M(p_i, p_j) = \mathbb{E}\Big[U_M(p_i, p_j)\Big] \tag{2}$$

where the expectation is w.r.t the joint p.m.f of (F_1, F_2). The payoff achieved by the SSPs can be similarly obtained. First, the random payoff of SSP-i is given by

$$U_i(p_i, p_j) = \begin{cases} F_i(P_i - c_S) & \text{if } P_i > P_j \\ (F_i + f_o)(P_i - c_S) & \text{if } P_i < P_j \\ (F_i + 0.5 f_o)(P_i - c_S) & \text{if } P_i = P_j \end{cases} \tag{3}$$

The average payoff can then be expressed as

$$\mathcal{U}_i(p_i, p_j) = \mathbb{E}\Big[U_i(p_i, p_j)\Big]. \tag{4}$$

Further, given that SSP-i has information about its flow-level (but not that of SSP-j), we define the following conditional payoffs: for $t \in \{\ell, h\}$

$$\mathcal{U}_{i,t}(p_i, p_j) \quad = \quad \mathbb{E}\Big[U_i(p_i, p_j)\Big| F_i = f_t\Big]. \tag{5}$$

In terms of the above conditional payoffs, the average payoff in (4) can be expressed as

$$\mathcal{U}_i(p_i, p_j) = \theta\, \mathcal{U}_{i,\ell}(p_i, p_j) + (1-\theta)\, \mathcal{U}_{i,h}(p_i, p_j). \tag{6}$$

Remark: While writing the above expressions it is assumed that $c_S \le P_i \le p_M$ ($i \in \{1,2\}$). The above assumption is natural because, otherwise (i) if $P_i > p_M$, the MNO will have no incentive to offload flow to SSP-i, or (ii) if $P_i < c_S$, SSP-i has no benefit in serving the offloaded flow.

D. Bayesian Game Formulation

Since the two SSPs are owned by different operators, the SSPs are naturally interested in choosing prices so as to maximizing their individual payoffs. The above problem can be studied using the framework of game theory. Specifically, since SSP-i lacks information about the flow F_j at SSP-j, we can formulate the *flexible-pricing problem* as a *Bayesian game* [9]. The correspondence between our model and the Bayesian game formulation can be established as follows:

- SSP-1 and SSP-2 constitute the **set of players**.
- The **set of states** is given by the following collection of flow-pairs: $\{(f_\ell, f_\ell), (f_\ell, f_h), (f_h, f_\ell), (f_h, f_h)\}$.

[1] In case of a tie, the overlap flow f_o is equally split among both the SSPs (as can be seen from the $P_i = P_j$ case in (3))

[2] For simplicity, here after we use the flexible notation (p_i, p_j) ($i \neq j$) to denote the price-vectors of both SSPs (instead of fixing these to (p_1, p_2)).

- The **probability distribution** over the set of states is given by θ^2, $\theta(1-\theta)$, $(1-\theta)\theta$ and $(1-\theta)^2$, respectively.
- The **set of types** of each SSP is simply the set of possible flow-level $\{f_\ell, f_h\}$.
- Given the type of SSP-i, the interval $[c_S, p_M]$ from which SSP-i can fix a price constitutes its **action space**.
- Finally, the payoffs $\mathcal{U}_{i,\ell}$ and $\mathcal{U}_{i,h}$ represents the **utility functions** of SSP-i given its respective types f_ℓ and f_h.

Using the framework of Bayesian games we finally characterize the solution in terms of Bayesian Nash equilibrium.

Definition 1 (Bayesian Nash Equilibrium): Price vectors (p_1^*, p_2^*) are said to constitute a Bayesian Nash equilibrium (BNE) if the following holds: for all $i \in \{1,2\}$ and $t \in \{\ell, h\}$ we have $\mathcal{U}_{i,t}(p_i^*, p_j^*) \geq \mathcal{U}_{i,t}(p_i, p_j^*)$ for all $p_i = (p_{i,\ell}, p_{i,h})$. Thus, unilateral deviation from a Bayesian Nash equilibrium does not benefit either of the SSPs. □

Remark: Technically, the above definition corresponds to that of a pure-strategy BNE. The mixed-strategy generalization is obtained by introducing probability distributions from which the SSPs can pick their respective price-vectors. Details of mixed-strategy BNEs will be discussed in Section IV. Meanwhile, in Section III we study the case of pure-strategy BNEs.

III. Pure Strategy BNEs

In this section we establish that there are no BNEs in pure-strategies. The results in this section are along the lines of the results in [5]. However, some caution is required as the flows F_i $(i = 1, 2)$ in our model are random which is unlike that in [5]. Hence, for completeness we review the results here. The details are as follows.

Suppose $F_i = f_t$ $(t \in \{\ell, h\})$ then SSP-i can accrue a guaranteed payoff of at least $u_t := f_t(p_M - c_S)$ by setting the price to maximum (i.e., $p_{i,t} = p_M$). However, by reducing the price (i.e., $p_{i,t} < p_M$), it is possible to achieve a higher payoff of $v_t(p_{i,t}) := (f_t + f_o)(p_{i,t} - c_S)$ (by acquiring the overlap flow f_o as well). Since v_t is decreasing in $p_{i,t}$ it follows that there exists a *threshold price* $\hat{p}_t$ such that for prices below $\hat{p}_t$ there is no incentive in acquiring the overlap flow; SSP-i is instead better-off by simply serving the monopoly flow at the maximum price. The value of $\hat{p}_t$ is obtained by solving $v_t(p_{i,t}) = u_t$, and is given by

$$\hat{p}_t = \frac{p_M f_t + c_S f_o}{f_t + f_o}. \tag{7}$$

Note that, since $c_S < p_M$ it follows that $c_S < \hat{p}_t < p_M$. In the following lemma we formally show that the prices below $\hat{p}_t$ are indeed dominated strategies.

Lemma 1: Consider a price-vector $p_i = (p_{i,\ell}, p_{i,h})$ of SSP-i $(i \in \{1,2\})$ such that $p_{i,t} < \hat{p}_t$ for some (or both) $t \in \{\ell, h\}$. Then p_i is strictly dominated.

Proof: Without loss of generality, suppose that $p_{i,\ell} < \hat{p}_\ell$. Let p_j be any price-vector of SSP-j. Then,

$$\begin{aligned}
\mathcal{U}_{i,\ell}(p_i, p_j) &\overset{(a)}{\leq} (f_\ell + f_o)(p_{i,\ell} - c_S) \\
&\overset{(b)}{<} (f_\ell + f_o)(\hat{p}_\ell - c_S) \\
&\overset{(c)}{=} f_\ell(p_M - c_S) \\
&\overset{(d)}{\leq} \mathcal{U}_{i,\ell}(\tilde{p}_i, p_j)
\end{aligned}$$

where $\tilde{p}_i = (p_M, p_{i,h})$ (i.e., $\tilde{p}_i$ is the price-vector obtained by replacing $p_{i,\ell}$ by p_M). In the above expression, (a) holds because the RHS represents the maximum payoff that SSP-i can possibly accrue at price $p_{i,\ell}$, (b) is due to the hypothesis $p_{i,\ell} < \hat{p}_\ell$, (c) simply follows from the definition of $\hat{p}_\ell$ in (7), and finally (d) holds because the actual payoff received at $\tilde{p}_i$ may include contribution from the overlap flow[3]. Thus, we see that p_i is strictly dominated by $\tilde{p}_i$. ■

Since the SSPs have no incentive to choose a dominated strategy, we can hence revise the range of allowable prices-vectors of the SSPs (i.e., the action set) to $[\hat{p}_\ell, p_M] \times [\hat{p}_h, p_M]$ (instead of $[c_S, p_M]^2$). With the assistance of the above result, we now state the following main theorem.

Theorem 1: There is no BNE in pure strategies for the *flexible-pricing game* developed in Section II-D.

Proof: Consider a generic pure-strategy price vector pair (p_1, p_2). We argue that (p_1, p_2) does not constitute a BNE in each of the following (exhaustive) cases:

(1) Suppose $p_{i,t} \in (\hat{p}_t, p_M]$ for some $i \in \{1,2\}$ and $t \in \{\ell, h\}$. Then, (a) if $p_{j,t} \in (p_{i,t}, p_M]$ $(j \neq i)$ then SSP-j can acquire the overlap flow (and thus increase its payoff) by choosing a price in $(\hat{p}_t, p_{i,t})$; (b) if $p_{j,t} = p_{i,t}$ then choosing a price less than, but arbitrarily close to $p_{i,t}$, SSP-j can benefit by acquiring the complete overlap flow f_o (instead of $0.5 f_o$ if it continues to use $p_{j,t}$); (c) Finally, if $p_{j,t} \in [\hat{p}_t, p_{i,t})$ then SSP-j is better-off choosing a price in $(p_{j,t}, p_{i,t})$.

(2) On the other hand, suppose $p_{i,t} = \hat{p}_t$ for all $i \in \{1,2\}$ and $t \in \{\ell, h\}$. Then, for $\tilde{p}_j = (p_M, p_M)$ we have

$$\begin{aligned}
\mathcal{U}_{j,t}(p_i, \tilde{p}_j) &= f_t(p_M - c_S) = (f_t + f_o)(\hat{p}_t - c_S) \\
&> \mathcal{U}_{j,t}(p_i, p_j)
\end{aligned}$$

where the last inequality simply follows because, using p_j, SSP-j would only acquire partial overlap flow. SSP-j can thus benefit by deviating to $\tilde{p}_j$. ■

With the above non-existence result in place, it is now natural to ask questions about mixed-strategy BNEs. Specifically, we are interested in determining *probability distributions* over the range of price-vectors (instead of individual price-vectors) that can constitute a solution to the SSPs' pricing problem. The details are presented in the following sub-section.

IV. Mixed Strategy BNEs

The definition of mixed-strategies involves allowing the SSPs to choose prices $p_{i,t}$ in a random fashion. Specifically, for $i \in \{1,2\}$ and $t \in \{\ell, h\}$, let $G_{i,t}$ denote the c.d.f of $p_{i,t}$, i.e., $G_{i,t}(p) = \mathbb{P}(p_{i,t} \leq p)$. We assume that $G_{i,t}$ is a distribution on the set of undominated prices $[\hat{p}_t, p_M]$ (since there is no rational in choosing any price less than $\hat{p}_t$; recall Lemma 1). Finally, we use $G_i = (G_{i,\ell}, G_{i,h})$ to denote a mixed-strategy of SSP-i.

[3] if $p_{j,t} = p_M$ for some (or both) $t \in \{\ell, h\}$; otherwise equality holds.

$$
\begin{aligned}
\mathcal{U}_{i,t}(p, G_j) &= \theta\Bigg(G_{j,\ell}(p) f_t (p - c_S) + \Big(1 - G_{j,\ell}(p)\Big)(f_t + f_o)(p - c_S)\Bigg) \\
&\qquad\qquad + (1-\theta)\Bigg(G_{j,h}(p) f_t (p - c_S) + \Big(1 - G_{j,h}(p)\Big)(f_t + f_o)(p - c_S)\Bigg) \qquad (8)\\
&= (f_t + f_o)(p - c_S) - \Big(\theta G_{j,\ell}(p) + (1-\theta) G_{j,h}(p)\Big) f_o (p - c_S). \qquad (9)
\end{aligned}
$$

Given a mixed strategy G_j of SSP-j, the payoff received by SSP-i for playing price p, given that $F_i = f_t$ ($t \in \{\ell, h\}$), can be written as in[4] (8). The payoff term associated with θ in (8) can be understood as follows. With probability θ the flow at SSP-j is f_ℓ. Then, further, with probability $G_{j,\ell}(p)$ SSP-j chooses a price less than p in which case SSP-i gets to serve only the monopoly flow f_t accruing a payoff of $f_t(p-c)$; on the other hand, with probability $(1 - G_{j,\ell}(p))$ the price set by SSP-j is greater than p in which case SSP-j receives a payoff of $(f_t + f_o)(p - c)$ by serving both monopoly as well as the overlap flows. The term associated with $(1-\theta)$ can be similarly understood. Simplifying (8) yields the simpler form in (9) for the expression of the payoff $\mathcal{U}_{i,t}$.

The expected payoff received by SSP-i if it uses the mixed-strategy $G_i = (G_{i,\ell}, G_{i,h})$ is given by

$$
\mathcal{U}_{i,t}(G_i, G_j) = \int_{\hat{p}_t}^{p_M} \mathcal{U}_{i,t}(p, G_j)\, dG_{i,t}(p) \qquad (10)
$$

where the integral in the above expression is understood as the *Riemann-Stieltjes integral* with respect to the distribution function $G_{i,t}$ [10]. We can now define mixed-strategy BNEs.

Definition 2 (Mixed-strategy BNE): A mixed-strategy pair (G_1^*, G_2^*) is said to constitute a mixed-strategy BNE for the flexible-pricing game if the following holds: for all $i \in \{1,2\}$ and $t \in \{\ell, h\}$ we have $\mathcal{U}_{i,t}(G_i^*, G_j^*) \geq \mathcal{U}_{i,t}(G_i, G_j^*)$ for all mixed-strategies G_i of SSP-i. A mixed-strategy BNE (G_1^*, G_2^*) is said to be *symmetric* if $G_i^* = G_j^* = G^*$. □

A. Deriving the Structure of a Mixed Strategy BNE

Obtaining mixed-strategy BNEs directly from the definition is difficult in general. However, there is an equivalent representation that can be used to compute mixed-strategy BNEs [9, Section 4.11, Proposition 140.1]. We state this result without proof (however, adapted to our notation) in the following.

Proposition 2 (Osborne 2004): Consider a mixed-strategy (G_i^*, G_j^*). For simplicity, let $u_{i,t}^* := \mathcal{U}_{i,t}(G_i^*, G_j^*)$. Then, (G_i^*, G_j^*) is a BNE if and only if, for each player $i \in \{1,2\}$ and state $t \in \{\ell, h\}$

- $G_{i,t}^*$ does not place any probability distribution on any p such that $\mathcal{U}_{i,t}(p, G_j^*) < u_{i,t}^*$.
- There exists no p such that $\mathcal{U}_{i,t}(p, G_j^*) > u_{i,t}^*$.

The above two conditions imply that $\mathcal{U}_{i,t}(p, G_j^*) = u_{i,t}^*$ for all p that lies in the domain of $G_{i,t}^*$, while $\mathcal{U}_{i,t}(p, G_j^*) < u_{i,t}^*$ for p that are outside its domain.

[4]For simplicity, in (8) we overload the notation $\mathcal{U}_{i,t}$ from (5) to also denote the payoff received by SSP-i in response to a mixed strategy of SSP-j.

Using the above proposition we proceed to identify mixed-strategy BNEs. Specifically, we compute a symmetric BNE[5] whose structure is as reported in the following theorem.

Theorem 3: The mixed-strategy pair (G^*, G^*) constitutes a symmetric BNE for the flexible-pricing game where $G^* = (G_\ell^*, G_h^*)$ is given by

$$
G_h^*(p) = \frac{(f_h + (1-\theta) f_o)}{(1-\theta) f_o}\ \frac{(p - q_h)}{(p - c_S)} \quad \text{for } q_h \leq p \leq p_M \qquad (11)
$$

$$
G_\ell^*(p) = \frac{(f_\ell + f_o)}{\theta f_o}\ \frac{(p - q_\ell)}{(p - c_S)} \quad \text{for } q_\ell \leq p \leq q_h. \qquad (12)
$$

The thresholds q_ℓ and q_h are computed as follows:

$$
q_h = \frac{p_M f_h + c_S (1-\theta) f_o}{f_h + (1-\theta) f_o} \qquad (13)
$$

$$
q_\ell = \frac{q_h (f_\ell + (1-\theta) f_o) + c_S \theta f_o}{f_\ell + f_o}. \qquad (14)
$$

Proof: Before proceeding to the details of the proof, it is useful to note that G_h^* and G_ℓ^* are indeed valid probability distributions on their respective domains. For instance, we see that G_h^* is increasing in p; also it can be verified (by direct substitution and simplification) that $G_h^*(q_h) = 0$ and $G_h^*(p_M) = 1$. Similarly, G_ℓ^* is also increasing in p, satisfying $G_\ell^*(q_\ell) = 0$ and $G_\ell^*(q_h) = 1$.

Now, the proof is essentially based on verifying the sufficient conditions in Proposition 2. Equivalently, we show that the payoff function $\mathcal{U}_{i,t}(p, G^*)$ is constant on the domain of G_t^*, and lower elsewhere. The details are as follows.

We begin by recalling the expression for $\mathcal{U}_{i,t}$ in (9) (adapted to the current case where $(G_i, G_j) = (G^*, G^*)$):

$$
\mathcal{U}_{i,t}(p, G^*) = \qquad (15)
$$
$$
(f_t + f_o)(p - c_S) - \Big(\theta G_\ell^*(p) + (1-\theta) G_h^*(p)\Big) f_o (p - c_S).
$$

For $t = \ell$, evaluating the above expression for $q_\ell \leq p \leq q_h$ we obtain (by noting that $G_h^*(p) = 0$ for p in the above range)

$$
\mathcal{U}_{i,\ell}(p, G^*) = (f_\ell + f_o)(q_\ell - c_S) =: u_\ell^*. \qquad (16)
$$

Thus, the payoff stays constant in the domain of G_ℓ^*. However, for $p < q_\ell$, since both $G_\ell^*(p) = G_h^*(p) = 0$, we obtain

$$
\mathcal{U}_{i,\ell}(p, G^*) = (f_\ell + f_o)(p - c_S) < u_\ell^*.
$$

[5]Determining non-symmetric BNEs (if any) is a scope for future work.

Finally, for $p \geq q_h$, substituting $G_\ell^*(p) = 1$ and $G_h^*(p)$ from (11), and simplifying yields

$$\mathcal{U}_{i,\ell}(p, G^*) = (f_\ell - f_h)p + \kappa_\ell \tag{17}$$

where $\kappa_\ell = \Big(f_h q_h - f_\ell c_S + (1-\theta) f_o (q_h - c_S)\Big)$ is a constant in p. Since $f_\ell < f_h$, it follows that the above expression is decreasing in p. Thus, for $p > q_h$ we have $\mathcal{U}_{i,\ell}(p, G^*) < \mathcal{U}_{i,\ell}(q_h, G^*) = u_\ell^*$.

Similarly, for $t = h$, evaluating the payoff expression in (15) for $q_h \leq p \leq p_M$ we obtain

$$\mathcal{U}_{i,h}(p, G^*) = (f_h + (1-\theta) f_o)(q_h - c_S) =: u_h^*. \tag{18}$$

On the other hand, for $p \leq q_h$ we obtain

$$\mathcal{U}_{i,h}(p, G^*) = (f_h - f_\ell)p + \kappa_h \tag{19}$$

where $\kappa_h = \Big(f_\ell q_\ell - f_h c_S + f_o (q_\ell - c_S)\Big)$. Since $f_h > f_\ell$ the above expression is increasing in p. Thus, for $p < q_h$ we obtain $\mathcal{U}_{i,h}(p, G^*) < \mathcal{U}_{i,h}(q_h, G^*) = u_h^*$. ■

B. Illustration and Discussion

In Fig. 1 we illustrate the structure of the BNE distributions G_h^* and G_ℓ^* for a numerical example (where $f_\ell = 5$, $f_h = 20$, $f_o = 10$ and $\theta = 0.6$). Also depicted in the figure are the respective payoff functions $\mathcal{U}_{i,\ell}$ and $\mathcal{U}_{i,h}$ (see the plots corresponding to the right-sided y-axis). The q_h and q_ℓ values (computed using (13) and (14)) are 8.5 and 5.5, respectively. These thresholds (marked on the x-axis) determine the respective domains of the BNE distributions G_h^* and G_ℓ^*. The payoff values $u_h^* = 180$ and $u_\ell^* = 67.5$ (computed using (18) and (16), respectively) are similarly marked on the right-sided y-axis. From Fig. 1 we make the following observations:

- We first note that the *domains are non-overlapping*. This condition is necessary as otherwise (due to the form of the payoff function in (15)) it would not be possible to satisfy the condition $\mathcal{U}_{i,t}(p, G^*) = u_t^*$ (i.e., a constant) for all p in the domain of G_t^*.
- Further, we also notice that *the distribution corresponding to the state of larger flow has a "greater" domain, and vice versa*. Specifically, the domain of G_h^* is $[q_h, p_M]$ which is greater than that of G_ℓ^* (which is $[q_\ell, q_h]$). The above requirement is necessary in the proof to show that the value of the payoff $\mathcal{U}_{i,t}(p, G_t^*)$ is lower outside the domain of G_t^*. For instance, from (17) we notice that the condition $f_\ell < f_h$ is critical to ensure that $\mathcal{U}_{i,\ell}(p, G^*) < u_\ell^*$ for $p \in (q_h, p_M]$. The above condition (i.e., $f_\ell < f_h$) is similarly required in (19) to argue that $\mathcal{U}_{i,h}(p, G^*) < u_h^*$ for p outside the domain of G_h^*.

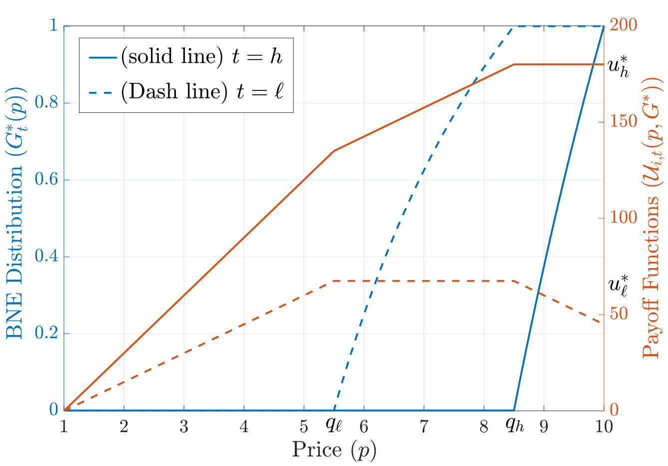


Fig. 1. Illustration of BNE distribution and payoff functions.

The above two observations, in fact, enabled us to derive the result in Theorem 3 as follows:

1) We first recognize that the domains of G_h^* and G_ℓ^* are of the form $[q_h, p_M]$ and $[q_\ell, q_h]$, respectively.
2) We then identify the forms of the distributions $G_t^*(p)$ ($t \in \{\ell, h\}$) that is necessary to ensure that the payoff $\mathcal{U}_{i,t}(p, G^*)$ remains constant over the respective domains. Expressions (11) and (12) are results of this process.
3) We finally calculate q_h and q_ℓ in an iterative fashion: we first obtain q_h by solving $G_h^*(p_M) = 1$; we then derive q_ℓ by solving $G_\ell^*(q_h) = 1$. The resultant expressions are as reported in (13) and (14).

The above procedure can be extended to scenarios comprising more than two flow levels. The details are presented in the following subsection.

C. Generalization to Scenarios with Multiple Flows

Consider a general model where $f_1 < f_2 < \cdots < f_n$ denote $n \geq 2$ different flow levels. The probability that F_i takes value f_t is given by θ_t (for $t \in [n]$ where $[n] := \{1, 2, \cdots, n\}$). The other aspects of the model remain unchanged. As a result, we have the following analog of expression (15):

$$\mathcal{U}_{i,t}(p, G^*) = (f_t + f_o)(p - c_S) - \sum_{s=1}^{n} \theta_s G_s^*(p) f_o (p - c_S)$$

where $G^* = (G_1^*, G_2^*, \cdots, G_n^*)$ represents a mixed-strategy BNE. Using the above payoff function into the 3-step procedure discussed earlier (and extending the procedure to $n \geq 2$ flow-levels), we obtain the following generalization of the result in Theorem 3:

Theorem 4: The mixed-strategy pair (G^*, G^*) constitutes a symmetric BNE where $G^* = (G_t^* : t \in [n])$ is given by

$$G_t^*(p) = \frac{\Big(f_t + \sum_{s=t}^{n} \theta_s f_o\Big)}{\theta_t f_o} \frac{(p - q_t)}{(p - c_S)} \text{ for } q_t \leq p \leq q_{t+1} \tag{20}$$

for $t \in [n]$. The thresholds q_t can be computed via. backward induction as follows: $q_{n+1} := p_M$ and

$$q_t = \frac{q_{t+1}\left(f_t + \left(1 - \sum_{s=1}^{t} \theta_s\right) f_o\right) + c_S \theta_t f_o}{f_t + \sum_{s=t}^{n} \theta_s f_o} \tag{21}$$

for $t = n, (n-1), \cdots, 1$. □

Proof: The proof is exactly along the lines of the proof of Theorem 3. We do not repeat the proof here for brevity. ■

It is easy to verify that for $n = 2$ the above result readily simplifies to the expressions in Theorem 3. Also, we note that the payoffs functions remain constant over the domain of the respective distributions. The value of the constant payoff u_t^* that is achieved by any p in the domain of G_t^* is given by

$$u_t^* := \left(f_t + \sum_{s=t}^{n} \theta_s f_o \right) (q_t - c_S). \tag{22}$$

We can again easily verify that the above expression is consistent with the expressions of u_ℓ^* and u_h^* in (16) and (18).

D. MNO Payoff

We finally compute the payoff received by the MNO when SSPs operate using the symmetric BNE $G^* = (G_\ell^*, G_h^*)$ derived in Theorem 3. For this, let us define $p_i = (p_{i,\ell}, p_{i,h})$ to be the random price-vector of SSP-i such that the distribution of $p_{i,t}$ is given by G_t^* (for $t \in \{\ell, h\}$). In other words, the distribution of p_i is given by G^*. Define $p_j = (p_{j,\ell}, p_{j,h})$ similarly, and assume that p_i and p_j are independent. Then, recalling (1) and (2) the average payoff of the MNO can be expressed as $u_M^* := \mathbb{E}\Big[\mathcal{U}_M(p_i, p_j)\Big]$.

Towards computing the above expectation, we define the conditional payoffs of the MNO as

$$u_M^*(t_i, t_j) := \mathbb{E}\Big[\mathcal{U}_M(p_i, p_j) \Big| F_i = t_i, F_j = t_j\Big]$$

for $t_i, t_j \in \{\ell, h\}$. Now, noting that $u_M^*(\ell, h) = u_M^*(h, \ell)$ (since both the SSPs are identical), we can write

$$\begin{aligned} u_M^* \quad = \quad & \theta^2 u_M^*(\ell,\ell) + 2\theta(1-\theta) u_M^*(\ell,h) \\ & + (1-\theta)^2 u_M^*(h,h). \end{aligned} \tag{23}$$

Recalling (1) and (2) again, the individual payoff terms in the above expression can be computed as follows: for $t \in \{\ell, h\}$

$$\begin{aligned} & u_M^*(t,t) = \\ & \quad 2f_t\Big(p_M - \mathbb{E}[p_{i,t}]\Big) + f_o\Big(p_M - \mathbb{E}\Big[\min\{p_{i,t}, p_{j,t}\}\Big]\Big) \end{aligned} \tag{24}$$

$$u_M^*(\ell,h) = \sum_{t \in \{\ell,h\}} f_t\Big(p_M - \mathbb{E}[p_{i,t}]\Big) + f_o\Big(p_M - \mathbb{E}[p_{i,\ell}]\Big).$$

Note that the above expressions have been simplified by replacing $\mathbb{E}[p_{j,t}]$ by $\mathbb{E}[p_{i,t}]$ for $t \in \{\ell, h\}$ (since both expectations are computed using the common distribution G_t^*). Also, we have made use of the observation $p_{i,\ell} < p_{j,h}$ (which follows from the domain of the distributions G_ℓ^* and G_h^*; recall Theorem 3) to simplify the $\min$-term in the second expression.

Expectation of $p_{i,t}$ can be computed using the identity

$$\mathbb{E}[p_{i,t}] = \int_0^\infty \mathbb{P}(p_{i,t} > p)dp = \int_0^\infty (1 - G_t^*(p))dp. \tag{25}$$

Similarly, since $p_{i,t}$ and $p_{j,t}$ are i.i.d it follows that

$$\mathbb{E}\Big[\min\{p_{i,t}, p_{j,t}\}\Big] = \int_0^\infty \Big(1 - G_t^*(p)\Big)^2 dp. \tag{26}$$

Using the above development in (23) and simplifying eventually yields (for brevity we have omitted the intermediate steps)

$$u_M^* = f_o(p_M - c_S) - \frac{2\theta(1-\theta)(p_M - c_S)(f_h - f_\ell)f_o}{f_h + (1-\theta)f_o}. \tag{27}$$

V. Flexible vs. Flat Pricing

The value of u_t^* in (22) denotes the payoff achieved by SSP-i ($i \in \{1, 2\}$) given that the respective flow-level is f_t. Thus, the average payoff is given by

$$u^* := \mathcal{U}_i(G^*, G^*) = \sum_{t=1}^{n} \theta_t u_t^*.$$

The above expression represents the payoff achieved by the SSPs under the flexible-pricing scheme, referred to as the *flexible-pricing payoff*. For $n = 2$, the flexible-pricing payoff is simply given by

$$u^* \quad = \quad \theta u_\ell^* + (1-\theta) u_h^* \tag{28}$$

where u_ℓ^* and u_h^* are as given in (16) and (18), respectively.

In this section we are interested in comparing the flexible-pricing payoff with the payoff achieved under a flat-pricing scheme (referred to as the *flat-pricing payoff*) where the SSPs are restricted to announce a single price, irrespective of the flow-level it may actually experience. This is in contrast to the flexible-pricing scheme where the SSPs are allowed to set different prices for different flow-levels it could experience.

In order to determine the flat-pricing payoff, we begin by establishing a correspondence between the flat-pricing scheme and the model studied by Li et al. in [5]. In [5], like in the flat-pricing scheme, the SSPs are asked to announce a single price. The model in [5] however assumes that the SSPs' flows (F_i) are deterministic, or in other words, there is a single possible flow-level. This is in contrast to our model where the flows F_i are assumed to be random, taking values from the set $\{f_\ell, f_h\}$ (or $\{f_t : t \in [n]\}$ in general). The above discrepancy between the two models can however be resolved by simply replacing the random flow in our model with the (deterministic) average flow $f_a = \theta f_\ell + (1-\theta) f_h$. Indeed, given the *flat-prices*[6] $(\overline{p}_i, \overline{p}_j)$ of the SSPs, the payoff received by SSP-i depends only on the average flow f_a as follows:

$$\overline{\mathcal{U}}_i(\overline{p}_i, \overline{p}_j) = \begin{cases} f_a(\overline{p}_i - c_S) & \text{if } \overline{p}_i > \overline{p}_j \\ (f_a + f_o)(\overline{p}_i - c_S) & \text{if } \overline{p}_i < \overline{p}_j \\ (f_a + 0.5 f_o)(\overline{p}_i - c_S) & \text{if } \overline{p}_i = \overline{p}_j \end{cases}$$

The above expression is identical to the payoff expression considered in [5]. Thus, leveraging the results from [5], we immediately identify the structure of the mixed-strategy BNE in the flat-pricing scheme.

Theorem 5 (Li et al. 2019): The mixed strategy $(\overline{G}^*, \overline{G}^*)$ constitutes a symmetric BNE for the flat-pricing scheme where

$$\overline{G}^*(p) = \frac{(f_a + f_o)}{f_o} \frac{(p - q_a)}{p - c_S} \quad \text{for } q_a \le p \le p_M. \tag{29}$$

[6] We use overline for all notations corresponding to the flat-pricing scheme. Also, note that $\overline{p}_i$ and $\overline{p}_j$ are scalar prices, which is unlike the case in the flexible-pricing scheme where these are vectors (e.g., $p_i = (p_{i,\ell}, p_{i,h})$)

The threshold q_a is given by

$$q_a = \frac{p_M f_a + c_S f_o}{f_a + f_o}. \quad (30)$$

Discussion: It is interesting to compare the above result with the form of the BNE for the flexible-pricing scheme in Theorem 3. For this, we first note that for $\theta = 0$ or $\theta = 1$, since only one of the flow-levels occur with probability 1, our model reduces to the scenario studied in [5]. Suppose $\theta = 0$ then, noting that $f_a = f_h$, and simplifying (11) and (13) we obtain $G_h^* = \overline{G}^*$ and $q_h = q_a$ (while the distribution G_ℓ^* degenerates). The case $\theta = 1$ similarly yields $G_\ell^* = \overline{G}^*$ and $q_\ell = q_a$. Thus, our result in Theorem 3 is a generalization of the above result by Li et al. in [5].

Now, for a given price p, the payoff received by SSP-i when SSP-j uses the mixed strategy $\overline{G}^*$ can be written as

$$\begin{aligned}
&\overline{\mathcal{U}}_i(p, \overline{G}^*) \\
&= \overline{G}^*(p) f_a (p - c_S) + \Big(1 - \overline{G}^*(p)\Big)(f_a + f_o)(p - c_S) \\
&= (f_a + f_o)(p - c_S) - \overline{G}^*(p) f_o (p - c_S) \\
&= (f_a + f_o)(q_a - c_S) \\
&= f_a (p_M - c_S)
\end{aligned}$$

for $p \in [q_a, p_M]$. Thus, we see that the payoff remains constant for p in the domain of $\overline{G}^*$. As a result, the flat-pricing payoff (i.e., the average payoff) is simply given by

$$\overline{u}^* := \overline{\mathcal{U}}_i(\overline{G}^*, \overline{G}^*) = f_a(p_M - c_S) \quad (31)$$

In the following lemma we establish the relation between the flexible and flat-pricing payoffs in (28) and (31).

Lemma 2: The difference in payoffs achieved by the SSPs in the flexible and the flat-pricing schemes is given by

$$(u^* - \overline{u}^*) = \frac{\theta(1-\theta)(p_M - c_S)(f_h - f_\ell) f_o}{(f_h + (1-\theta) f_o)}. \quad (32)$$

Proof: Substituting the expressions of q_h, q_ℓ and q_a (from (13), (14) and (30), respectively) into $(u^* - \overline{u}^*)$, and simplifying yields the above result. Since the steps are straightforward, we do not present the details here for brevity. ■

Discussion: Since $p_M > c_S$ and $f_h > f_\ell$, from (34) it follows that $u^* \geq \overline{u}^*$, with equality if and only if $\theta = 0$ or $\theta = 1$. Thus, the SSPs can achieve a higher payoff in the flexible-pricing scheme than if flat-pricing were to be implemented. Although we expect similar result for the general case comprising more than two flow-levels (i.e., $n \geq 2$), we leave the details of the generalization to future work. Here, we instead conduct a numerical study to demonstrate the efficacy of the flexible-pricing scheme for the general case. Details are available in Section VI.

Finally, we compute the payoff received by the MNO under the flat-pricing scheme. Analogous to the conditional payoff term $u_M^*(t,t)$ in (24), the MNO's payoff in the flat-pricing scheme can be written as

$$\overline{u}_M^* := 2 f_a \Big(p_M - \mathbb{E}[\overline{p}_i]\Big) + f_o \Big(p_M - \mathbb{E}\Big[\min\{\overline{p}_i, \overline{p}_j\}\Big]\Big)$$

where $\overline{p}_i$ and $\overline{p}_j$ are i.i.d random prices with their common c.d.f given by $\overline{G}^*$ in (29). Substituting the expectation terms (which can be computed using the identities in (25) and (26)) in the above expression and simplifying yields the following simple form for the payoff expression

$$\overline{u}_M^* = f_o(p_M - c_S). \quad (33)$$

Analogous to the result in Lemma 2, we have the following:

Lemma 3: The difference in payoffs achieved by the MNOs in the flexible and the flat-pricing schemes is given by

$$(u_M^* - \overline{u}_M^*) = -\frac{2\theta(1-\theta)(p_M - c_S)(f_h - f_\ell) f_o}{(f_h + (1-\theta) f_o)}. \quad (34)$$

Proof: The proof easily follows by recalling the expressions of u_M^* and $\overline{u}_M^*$ from (27) and (33), respectively. ■

Discussion: From the above result we see that MNO achieves a lower payoff in the flexible-pricing scheme. However, comparing the above result with that in Lemma 2, we see that the amount of loss in payoff incurred by the MNO is equal to the total gain in payoff achieved by both SSPs. Thus, the net-payoff in the system is conserved when moving from flat to flexible-pricing scheme. Further, from mechanism design point-of-view, the MNO is in fact not at loss in the flexible-pricing scheme as its payoff in (27) remains non-negative inspite of being a neutral entity who sets up the game between the SSPs (by first fixing the price p_M, and then instructing the SSPs to announce their flexible prices). Formally, the flexible-pricing scheme is *incentive compatible* as all entities achieve non-negative payoffs.

VI. Numerical Work

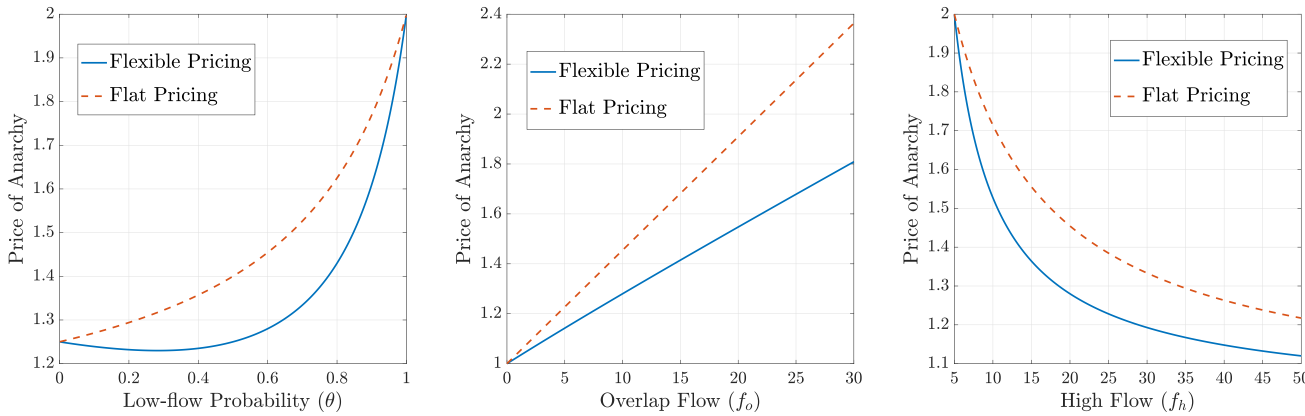
In this section we will compare the performances of the flexible and the flat-pricing schemes. Instead of comparing the raw payoffs (u^* and $\overline{u}^*$) we choose to use the *price-of-anarchy (PoA)* metric that takes into account the social optimal payoff that the SSPs could achieve it they choose to cooperate with one another while setting the prices. Formally, the PoA, for instance, for the flexible-pricing scheme is defined as follows:

$$PoA = \frac{\text{Social Optimal Payoff}}{\text{Payoff at BNE}} = \frac{u_{opt}}{u^*}. \quad (35)$$

where $u_{opt} = (f_a + 0.5 f_o)(p_M - c_S)$, which is the maximum payoff that each SSP can accrue if both SSPs were to choose the price-vector $p = (p_M, p_M)$[7]. From the above definitions it follows that $PoA \geq 1$ with a lower value being more preferable (as is would imply that the payoff at the BNE is closer to the optimal payoff). Similarly, the PoA for the flat-pricing scheme is given by $\overline{PoA} = u_{opt}/\overline{u}^*$.

In Fig. 2 we present the results of our PoA study. We first fix the values of the following parameters: $c_S = 1$, $p_M = 10$ and $f_\ell = 5$. The other parameters θ, f_o and f_h are then varied to obtain the respective plots. For instance, in Fig. 2(a) we depict PoA vs. θ for $f_o = 10$ and $f_h = 20$. The plots in Fig. 2(b) and 2(c) are similarly obtained by varying f_o and

[7]However note that (p, p) does not constitute a BNE, and hence is not considered a rational solution to the SSPs' pricing problem.



(a) (b) (c)

Fig. 2. PoA comparison of the flexible and flat-pricing schemes (where $c_S = 1, p_M = 10$ and $f_\ell = 5$) (a) PoA vs. θ (for $f_h = 20$ and $f_o = 10$), (b) PoA vs. f_o (for $\theta = 0.6$ and $f_h = 20$), and (c) PoA vs. f_h (for $\theta = 0.6$ and $f_o = 10$).

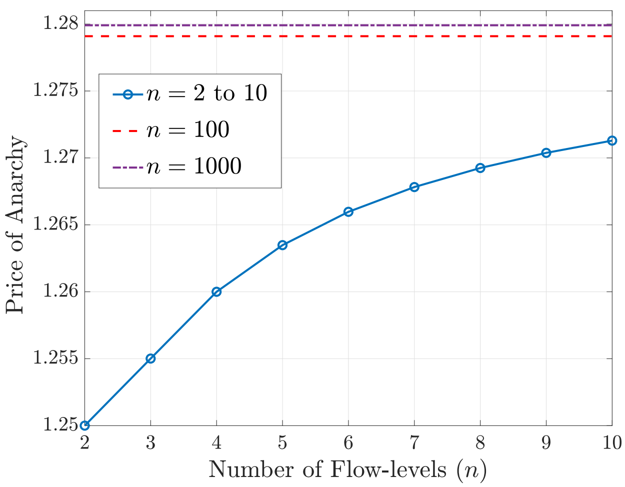


Fig. 3. PoA of the general model comprising $n \geq 2$ flow-levels

f_h, respectively, by fixing the values of the other parameters. Inspecting the plots in Fig. 2 we conclude that the overall performance of the flexible-pricing scheme is superior to that of the flat-pricing scheme.

In Fig. 3 we study the effect of increasing n (number of flow-levels) on the PoA of the flexible-pricing scheme. As before we fix $c_S = 1$ and $p_M = 10$, while we set $f_o = 10$. Then, for a given $n \geq 2$ we choose n equally spaced flow-levels f_t ($t \in [n]$) in the interval $f_1 = 5$ to $f_n = 20$. All the n flows are assumed equally likely so that $\theta_t = 1/n$ for all $t \in [n]$. We make the following observations from Fig. 3: (i) First, we note that PoA is increasing with n, implying that there is no benefit in providing a detailed distribution of the flows to the SSPs. (ii) The increments in PoA are however reducing, thus implying that the PoA may eventually converge for larger values of n (from Fig. 3, notice the marginal increment in the PoA value as n is increased from 100 to 1000).

Finally, we compare the PoA of the flexible-pricing scheme in Fig. 3 with the corresponding PoA of $\overline{PoA} = 1.4$ incurred by the flat-pricing scheme. Since the flat-pricing payoff depends only on the average flow f_a ($= 12.5$ for all n) the $\overline{PoA}$ value does not vary with n. From Fig. 3 we notice that the PoA of the flexible-pricing scheme (including $n = 100$ and 1000) always remains lower than the PoA incurred by the flat-pricing scheme. The flexible-pricing scheme is thus also efficient in the general scenario comprising more than 2 flow-levels.

VII. Conclusion

We proposed a flexible-pricing scheme for the problem of mobile data offloading in scenarios where the flows (i.e., offloaded traffic) are random. Formulating the problem as a Bayesian game, we derived results illustrating the structure of a symmetric mixed-strategy BNE (Theorem 3 and 4). We also conducted a study to compare the flexible-pricing scheme with the traditional flat-pricing scheme. The efficacy of the flexible-pricing scheme over the flat-pricing scheme (from SSPs point-of-view) was also demonstrated via. a numerical study.

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