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# Local Construction of Connected and Plane Spanning Subgraphs under Acyclic Redundancy

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***Abstract*—Plane graphs play a major role for local routing and some other local network protocols in wireless communication. With such local algorithms each node requires information about its neighborhood only. It is assumed that nodes are deployed on the plane and each node knows its position in a given coordinate system. An arbitrary graph drawn on the plane can be transformed into a plane spanning subgraph by deleting edges. However, to assure connectivity at the same time some additional structural graph properties are required. Current graph classes that assure the existence of connected plane spanning subgraphs require assumptions, that are not very likely to hold for wireless network structures. In this work we develop the acyclic redundancy condition. This is a novel graph class with only one property that assures the existence of a connected plane spanning subgraph. Furthermore, we describe local algorithms that construct a connected plane spanning subgraph for graphs satisfying the acyclic redundancy condition. With numerical studies we confirm that the acyclic redundancy condition is a more realistic condition than existing graph classes that were required so far to construct connected plane spanning subgraphs.**

***Index Terms*—plane subgraph, local algorithm, log-normal-shadowing, stochastic analysis, numerical evaluation**

## I. INTRODUCTION

Many distributed local algorithmic solutions for large scale wireless networks like unicast [1]–[3], multicast [4], [5], geocast [6], anycast [7], mobicast [8], broadcast [9], [10], void and boundary detection [11], distributed data storage [12], [13], tracking of mobile objects [14], localized address auto-configuration [15], or coordination of mobile sensors [16] require the network graph to be drawn on a plane without intersecting edges. Here local means that a global network wide objective is achieved just by local node decisions based on neighborhood information.

Original methods for finding such plane graphs (i.e. graphs without intersecting edges) as described in [1], [2], [17], [18] require the network to be given as a unit disk graph (UDG), i.e. two nodes are connected if and only if their Euclidean distance is less or equal than a given network wide unique unit disk radius. In general, the unit disk assumption does not hold for wireless networks. Thus, subsequent work dealt with algorithm modifications and model extensions such that local construction of intersection free drawings can locally be found beyond the unit disk graph assumption. This includes, methods for quasi unit disk graphs [19], [20] which is a model extension of unit disk graphs, and as well the recently studied class of graphs satisfying redundancy [21], [22] and coexistence [23] property. Redundancy assures that intersections can be found locally while coexistence supplementally assures existence of a connected plane spanning subgraph [24]. Recently it was found that such an intersection free spanning subgraph can also be found by local rules [25].

In this paper we advance the state of the art on the question how far graph assumptions can be weakened such that realistic wireless networks are closely reproduced while the graph properties assure design of local algorithms with provable correctness guaranteed. Compared to the related work we develop a property which is an advancement of the redundancy property. Proper redundancy guarantees the existence of edges to detour and hence delete intersecting edges. However there are certain graph structures preventing the existence of a connected and plane spanning subgraph (for details see [24]). To rule out those scenarios the coexistence property was used. However, the coexistence property demands much more edges to exist than needed to avoid those so called redundancy cycles. Hence, we are searching for alternative graph properties compensating coexistence. We introduce two variants of a novel graph class with just one property, the (basic resp. extended) acyclic redundancy graph condition. Redundancy follows only from basic acyclic redundancy. Coexistence follows from none of them, so the acyclic redundancies can be regarded as a new approach for graph conditions that assure existence of connected plane spanning subgraphs. We present global and local algorithms which construct plane spanning subgraphs and prove their correctness. Moreover, we analyze how far these properties can be assumed to hold in realistic wireless networks described by the log-normal shadowing model.

The remainder is structured as follows. In the next section we introduce the required graph concepts and the underlying wireless network model. In section III we define the two versions of acyclic redundancy and present algorithms to construct a plane spanning subgraph under those conditions. Furthermore, we prove correctness of the algorithms and examine the relation of the different graph properties. In section IV

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we derive, by means of stochastic geometry, expressions to compute how far the novel properties can be assumed to hold in random geometric graph models. The result is then used in section V where we show numerically that under the log-normal shadowing model extended acyclic redundancy is more likely to be satisfied than proper redundancy (with or without coexistence) in relevant settings. We discuss under which parameter settings the property can be assumed to hold closely to $100\%$ and when it can no longer be assumed to be likely. Furthermore, similar it was applied in [26], we discuss the concept of artificially bounding the communication range to improve the probability that acyclic redundancy holds. We conclude our work in section VI where we judge how far the model may still be relaxed while still supporting local solutions and where limitations on such local solutions have to be expected.

## II. Assumptions and Definitions

We consider *embedded* undirected graphs $G = (V, E)$ where each vertex $v \in V$ is a point in the $\mathbb{R}^2$ plane and each edge $uv \in E$ a straight line connecting $u$ and $v$. A path between two vertices $u, v \in V$ is a sequence $w_0, \ldots, w_k \in V$ with $u = w_0$, $v = w_k$ and pairwise disjoint edges $w_i w_{i+1} \in E$. We term $k$ the length of such a path. We call a path with $w_0 = w_k$ *polygon* of length $k$. A polygon of length 3 is a *triangle*. A polygon of length $4$ is a *quadrilateral*. A graph is called a *plane graph* if each pair of its edges have at most one end point in common. A pair of edges having another point $p$ in common are said to *intersect* each other.

We say $F = (V, E')$ is a *(spanning) subgraph* of $G$ if $E' \subseteq E$. We say a *graph is connected* if for all $u, v \in V$ there is a path from $u$ to $v$. We say $F = (V, E')$ is a *connectivity-preserving subgraph* of $G$ if the connected components of $F$ and $G$ coincide.

Since we do not consider any other subgraphs we drop the spanning property from the notation.

The $k$-hop neighborhood of a vertex $v$ refers to all paths of length at most $k$ starting in $v$. The vertices contained in these paths are termed $k$-hop neighbors of $v$. The $k$-hop neighbors of a vertex $v$ are termed $N_k(v)$. The edges connecting a vertex to its 1-hop neighbor are termed *outgoing edges*. Edges sharing one endpoint are called each others *edge-*$1$*-hop neighbors*.

In this work we are given a connected graph $G$ and the goal is to construct a connected and plane subgraph of $G$ by *local rules*. This means a distributed algorithm on each vertex $v$ which eventually decides for each outgoing edge of $v$ if it has to be kept or removed. In this work each vertex needs to know its 2-hop neighborhood. Based on that information without any further message exchange it decides immediately which of its outgoing edges are removed.

Note that the algorithms and the proved results work as well for not necessarily connected graphs and a connectivity-preserving subgraph of a connected graph is connected again. Thus, we can generalize the problem setting to construct a connectivity-preserving plane subgraph.

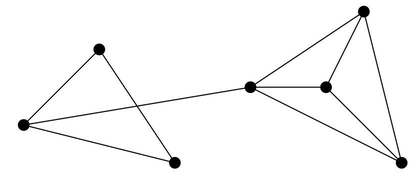

Fig. 1. Example of an RCG graph, satisfying redundancy on the left and coexistence on the right.

A graph satisfies the *(proper) redundancy property* (PR) if for all intersecting edges $uv$ and $wx$ at least one edge end point is 1-hop neighbor of the other three ones (see Fig. 1). A graph satisfies the *coexistence property* if for all triangles $u, v, w$ with an interior vertex $x$, $x$ is 1-hop neighbor of $u$, $v$ and $w$. A graph satisfying both properties is called *redundancy and coexistence graph (RCG)*.

Furthermore, we assume to have a strict total order $\prec$ on the edge set $E$, i.e. for all $e \neq f \in E$ either $e \prec f$ or $e \succ f$. A locally testable order can be obtained by edge weights $c$. We use the induced decreasing total order $\preceq_c$ given by $e \preceq_c f$ iff. $c(e) \leq c(f)$. We call $f$ *c-heavier* than $e$. To get a strict total order one has to break ties, i.e. define a total order on edges with the same weight. That can be done by e.g. using the endpoints' positions.

To study the question how far certain structural properties can be assumed to hold with high probability in real wireless networks, we consider the following commonly used random graph model.

For the analysis we use a *homogeneous Poisson point process* (PPP) of intensity $\lambda$ as vertex set. The number of vertices in an area of size $A$ is a Poisson distributed random variable with mean $\lambda A$. Furthermore, the number of vertices in two disjoint areas are independent.

On this vertex set we consider a random geometric graph model. Given a monotonically decreasing connection function $p : (0, \infty) \to [0, 1]$ two nodes with distance $d$ are connected by an edge with probability $p(d)$. The existence of the edges are assumed to be independent. Note, as also required in [26] $p$ has to guarantee that edge lengths have finite expectation and that the integral expressions for edge intersections converge (see section IV).

Note that with probability 1 no two vertex pairs in a homogeneous PPP have the same distance. Hence the edge lengths are unique with probability 1 and an ordering just based on edge lengths is a well defined total ordering.

The *log-normal shadowing* model is widely known as a realistic model describing wireless network graphs. The received signal strength $P_{RX}(d)$ in dBm of a transmission over distance $d$ is given by

$$P_{RX}(d) = c - 10\alpha \log_{10}(d/d_0) + X_\sigma \tag{1}$$

where $d_0 > 0$ is the reference distance, $c$ the power received at reference distance, $\alpha > 1$ the path loss coefficient, and $X_\sigma$ a Gaussian random variable with mean 0 and standard deviation

$\sigma > 0$. By [27] we can reformulate the probability that an edge of length $d$ exists to be the following connection function

$$p_{LNS}(d) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{10}{\sqrt{2}}\frac{\alpha}{\sigma}\log\left(\frac{d}{r_0}\right)\right) \tag{2}$$

with average communication distance $r_0$.

## III. Construction of a connectivity-preserving plane subgraphs under acyclic redundancy

### A. Acyclic Redundancy

In the subsequent sections all properties and algorithms are defined on (injectively) weighted graphs. Since a strict total edge order can also be transferred to a weight function, everything holds as well for any arbitrary strict total ordering of the edges.

**Definition 1.** *A weighted embedded graph $(G, c)$ satisfies the* basic acyclic redundancy property (BAR) *if for each pair of intersecting edges $uv, wx$ one of the intersecting edges is the $c$-heaviest edge of a triangle in $G$ within $\{u, v, w, x\}$.*

An example with Euclidean distance as weight function is shown in Fig. 2.

**Definition 2.** *A weighted embedded graph $(G, c)$ satisfies the* extended acyclic redundancy property (EAR) *if for each pair of intersecting edges $uv, wx$ one of the intersecting edges is the $c$-heaviest edge of a triangle or quadrilateral in $G$ within $\{u, v, w, x\}$.*

Examples are illustrated in Fig. 2, Fig. 5 and Fig. 6. If we do not further specify whether basic or extended acyclic redundancy is regarded, we just say acyclic redundancy. We term graphs satisfying the acyclic redundancy ARG.

**Definition 3.** *An embedded graph $G$ satisfies the* basic (extended) Euclidean redundancy property *if $(G, d_2)$ satisfies the basic (extended) acyclic redundancy property, where $d_2$ is the Euclidean distance between two vertices.*

Note that basic acyclic redundancy implies both, proper and extended acyclic redundancy.

Proper redundancy is implied because the demanded triangle requires two adjacent edges from the possible quadrilateral, which is spanned by the intersection, to exist. This is an equivalent formulation of proper redundancy.

Extended acyclic redundancy follows from basic acyclic redundancy because the triangle from basic acyclic redundancy is a polygon demanded for the extended property.

Note that no further implication holds in general. Thus, neither EAR and PR are implying each other (see Fig. 4 and Fig. 5), nor is BAR implied by one of them nor their conjunction (Fig. 6).

The already studied properties and the properties defined in this work are related as follows (see Fig. 3). Compared to the UDG property, the acyclic redundancy is a property which is more likely to hold. It directly implies basic acyclic redundancy. Therefore the unit disk property also implies the extended acyclic redundancy property. The other implication does not hold in general, since for a pair of intersecting edges, not all redundancy edges that are shorter than one of the intersecting edges have to exist (see Fig. 2).

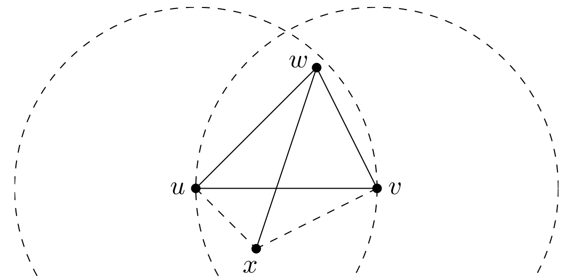


Fig. 2. Example for a graph that satisfies the Euclidean acyclic redundancy which is not a UDG.

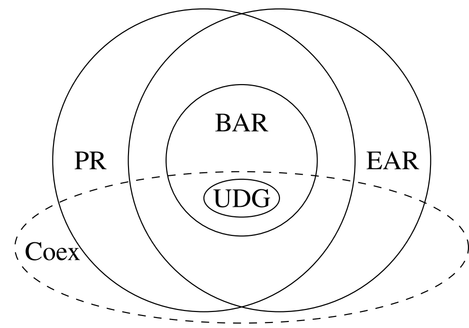


Fig. 3. Venn diagram depicturing the relations of the graphs satisfying the different redundancies, the unit-disk-graph-property and the coexistence property.

### B. Algorithmic constructions of plane subgraphs

In this section we will show that graphs satisfying basic or extended acyclic redundancy can be transformed into a connectivity-preserving plane subgraph that contains no triangles.

**Theorem 1.** *Algorithm 1 outputs a connectivity-preserving plane subgraph under basic acyclic redundancy.*

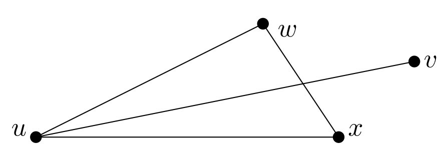


Fig. 4. Example for an intersection that satisfies the proper redundancy property, but not the extended acyclic redundancy property.

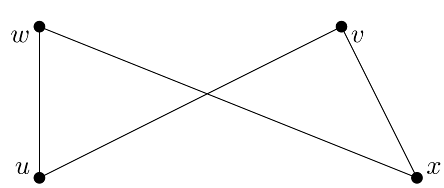


Fig. 5. Example for an intersection that satisfies the extended acyclic redundancy property, but not the proper redundancy property.

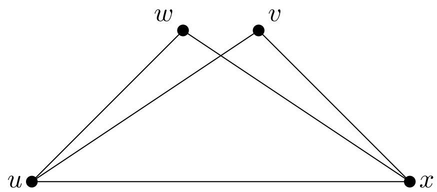


Fig. 6. Example for an intersection that satisfies proper and extended acyclic redundancy but not basic acyclic redundancy.

**Algorithm 1** Global algorithm for basic acyclic redundancy

$F \leftarrow E$ (* initialize the result *)
$W \leftarrow E$ (* set up the working set *)
**while** $W \neq \emptyset$ **do**
  choose $c$-heaviest $uv \in W$
  remove $uv$ from $W$
  **if** $\exists uw, vw \in F$ **then**
    remove $uv$ from $F$
  **end if**
**end while**
G' = (V,F)

*Proof.* The connectivity-preserving follows directly from the formulation of the algorithm, since an edge $uv$ can only be removed if in the so far constructed subgraph there is still an alternative path of the edges $uw$ and $wv$ left.

For the planarity, assume there is an intersection between the edges $uv$ and $wx$ left after termination of the algorithm. However, because of the basic acyclic redundancy property there would have been two redundancy edges, w.l.o.g. $uw$ and $ux$ that have been shorter than $wx$. Thus, $wx$ would have been removed. Since this holds for all intersections, the resulting graph G' is intersection free. □

**Algorithm 2** Distributed edge-1-hop algorithm from the view of an edge $e = uv$

$F \leftarrow E$ (* initialize the result *)
**if** $\exists uw, vw \in E | c(uw), c(vw) < c(uv)$ **then**
  remove $e$ from $F$
**end if**
G' = (V,F)

**Theorem 2.** *Algorithm 1 and Algorithm 2 yield the same output for the same input.*

*Proof.* Suppose the results differ from each other. Let $e$ be the $c$-heaviest edge which is contained in exactly one of the two resulting graphs. At the moment, when $e$ is decided in the global version all triangles from the initial graph containing $e$ and a longer edge are already canceled. The heaviest edge has already been processed and has been canceled because of the triangle. Obviously all triangles of the initial graph where $e$ is the longest edge are still there, since all the other (shorter) edges have not been processed so far. So at that point the triangles containing $e$ are exactly those (from the initial graph) where $e$ has been the longest edge. Hence all triangles relevant for the decision of $e$ in the global algorithm are considered in the local one and vice versa. Hence the decision on $e$ will be identical for both algorithms, which contradicts the assumption. □

Note that the algorithm is edge-1-hop local, i.e. each edge only needs the information of the edge-1-hop-neighbors. If one wants to formulate it as a procedure for vertices one needs 2-hop information (i.e. information about the neighbors of the neighbors) to detect triangles. However, if vertices have that information we can detect quadrilaterals to detour edges with a path of length 3. This leads to the following versions of the algorithms.

**Algorithm 3** Global algorithm for extended acyclic redundancy

$F \leftarrow E$ (* initialize the result *)
$W \leftarrow E$ (* set up the working set *)
**while** $W \neq \emptyset$ **do**
  choose $c$-heaviest $uv \in W$
  remove $uv$ from $W$
  **if** $\exists uw, vw \in F \vee \exists uw, vx, wx \in F$ **then**
    remove $uv$ from $F$
  **end if**
**end while**
G' = (V,F)

**Algorithm 4** Distributed 2-hop algorithm from the view of an vertex $u$

$F \leftarrow E$ (* initialize the result *)
**for** $v \in N_1(u)$ **do**
  **if** $\exists uw, vw \in E | c(uw), c(vw) < c(uv) \vee \exists uw, vx, wx \in E | c(uw), c(vx), c(wx) < c(uv)$ **then**
    remove $e$ from $F$
  **end if**
**end for**
G' = (V,F)

**Theorem 3.** *Algorithm 3 outputs a connectivity-preserving plane subgraph under extended acyclic redundancy.* □

**Theorem 4.** *Algorithm 4 outputs the same graph as Algorithm 3.* □

The proofs are analogous to the respective proofs of the basic versions.

**Theorem 5.** *The resulting graphs of all four algorithms still contain an minimum spanning tree (MST)*

*Proof.* Theorem 2 and Theorem 4 state that it is enough to show that Algorithms 1 and 3 yield graphs that contain an MST. Since Algorithm 1 deletes only a subset of the edges deleted by Algorithm 3, the only implication left to show is that Algorithm 3 yields a graph $G'$ that contains an MST. The reverse-delete-algorithm of Krsukal [28] is quite similar to

Algorithms 1 and 3. It removes edges $e$ from $F$ in decreasing order if $F \setminus e$ does not contain more partitions than $F$. An equivalent formulation of the algorithm is, that an edge is deleted if and only if it is the heaviest edge of any cycle in the initial graph (assuming injective weights). Since Algorithm 3 removes an edge only if it is the heaviest edge of a cycle of length at most four, the resulting graph from the reverse-delete-algorithm is a subset of the resulting graph of Algorithm 3. Therefore an MST is contained in the resulting graph of Algorithm 3. $\square$

**Remark 1.** *The resulting graph of Algorithm 1 does not contain any cycles of length* 3*. Hence it is an MST if* $G$ *does not contain a cycle of length* 4 *or more. The resulting graph of Algorithm 3 does not contain cycles of length three or four. Thus, it is an MST if* $G$ *does not contain a cycle of length* 5 *or more.*

**Remark 2.** *Another characteristic of the resulting graph of one of the four algorithms is, that* $G'$ *is* 3*-colorable if* $G$ *is triangle-free since* $G'$ *is a plane graph and therefore also a planar graph without triangles.*

## IV. Stochastic Analysis

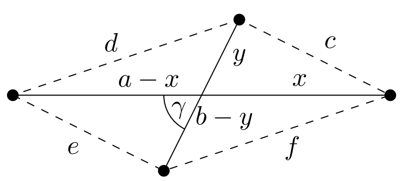


Fig. 7. Illustration of the parameterization of an intersection.

From this section onwards we fix the Euclidean distance as edge weights. Therefore we call the weight of an edge its length as well. Furthermore we assume that the lengths are unique.

In [26] we developed the following probability density function for intersections in random geometric graphs on a homogeneous PPP.

$$f_{int}(a,b,x,y,\gamma) = \frac{1}{N_{int}} ab \sin\gamma p(a)p(b) \mathbb{1}_{\substack{b,x\in(0,a)\\ y\in(0,b)\\ \gamma\in(0,\pi)}} \tag{3}$$

with $a$, $b$ being the lengths of the intersecting edges, $\gamma$ the common angle and $x$, $y$ the intersection points on the respective edge. $N_{int}$ is a normalization constant. Furthermore we can assume $a < b$.

For notational simplification let $[z]$ be the event that edge $z$ exists and $[z < w]$ be the event that $z$ exists and it is shorter than $w$.

For proper redundancy we get the formulation:

$$([c] \vee [e]) \wedge ([d] \vee [f])$$

The basic acyclic redundancy can be formulated as:

$$\begin{aligned} &([c < a] \wedge [d < a]) \vee ([e < a] \wedge [f < a]) \\ \vee &([c < b] \wedge [f < b]) \vee ([d < b] \wedge [e < b]) \end{aligned}$$

We can formulate the probability of basic Euclidean acyclic redundancy for a given setting as the probability that at least one of the four clauses holds. However, since more than one of the clauses can be satisfied simultaneously and they are not independent of each other a separation into disjoint events is necessary. A possible separation is

$$\begin{aligned} &([c < a] \wedge [d < a]) \\ \vee &([e < a] \wedge [f < a] \wedge \neg([c < a] \wedge [d < a])) \\ \vee &([c < b] \wedge [f < b] \wedge \neg[d < a] \wedge \neg[e < a]) \\ \vee &([d < b] \wedge [e < b] \wedge \neg[c < a] \wedge \neg[f < a]) \end{aligned}$$

Since the existences of the edges $c, d, e, f$ are independent of each other once the lengths are set, we can calculate the probability to satisfy a clause by multiplying the corresponding probabilities. Since the events are disjoint we can just add them up which yields the probability that two intersecting edges satisfy the basic acyclic redundancy to be

$$\begin{aligned} p_{bar} = &p_a(c)p_a(d) + p_a(e)p_a(f)(1 - p_a(c)p_a(d)) \\ &+ p_b(c)p_b(f)(1 - p_a(d))(1 - p_a(e)) \\ &+ p_b(d)p_b(e)(1 - p_a(c))(1 - p_a(f)) \end{aligned}$$

with $p_w(z) = p(z)\mathbb{1}_{z \le w}$for a given setting $(a, b, x, y, \gamma)$

$$\begin{aligned} c &= \sqrt{x^2 + y^2 - 2xy\cos\gamma} \\ d &= \sqrt{(a-x)^2 + y^2 + 2(a-x)y\cos\gamma} \\ e &= \sqrt{(a-x)^2 + (b-y)^2 - 2(a-x)(b-y)\cos\gamma} \\ f &= \sqrt{x^2 + (b-y)^2 + 2x(b-y)\cos\gamma} \end{aligned}$$

Translating the indicator function of the probability density (3) into the integration bounds we get for the probability of a random intersection satisfying basic acyclic redundancy

$$\int_0^\infty \int_0^a \int_0^\pi \int_0^a \int_0^b (p_{bar} f_{int})(a, b, x, y, \gamma) \mathrm{d}y \mathrm{d}x \mathrm{d}\gamma \mathrm{d}b \mathrm{d}a$$

For the extended acyclic redundancy we get the following formulation:

$$\begin{aligned} &([c < a] \wedge [d < a]) \vee ([e < a] \wedge [f < a]) \\ \vee &([c < b] \wedge [f < b]) \vee ([d < b] \wedge [e < b]) \\ \vee &([c < a] \wedge [e < a]) \vee ([d < a] \wedge [f < a]) \end{aligned}$$

Again a separation into disjoint events is necessary and it can be formulated as:

$$\begin{aligned} &([c < a] \wedge [d < a]) \\ \vee &([e < a] \wedge [f < a] \wedge \neg([c < a] \wedge [d < a])) \\ \vee &([c < b] \wedge [f < b] \wedge \neg[d < a] \wedge \neg[e < a]) \\ \vee &([d < b] \wedge [e < b] \wedge \neg[c < a] \wedge \neg[f < a]) \\ \vee &([c < a] \wedge [e < a] \wedge \neg[d < a] \wedge \neg[f < a]) \\ \vee &([d < a] \wedge [f < a] \wedge \neg[c < a] \wedge \neg[e < a]) \end{aligned}$$

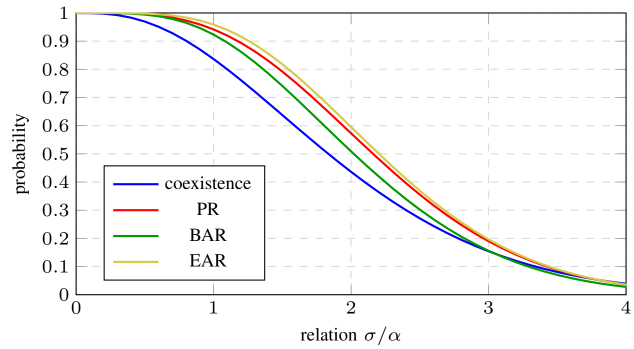


Fig. 8. Probability of proper (PR), basic acyclic (BAR), extended acyclic redundancy (EAR) and coexistence over $\sigma/\alpha$ for $R = \infty$.

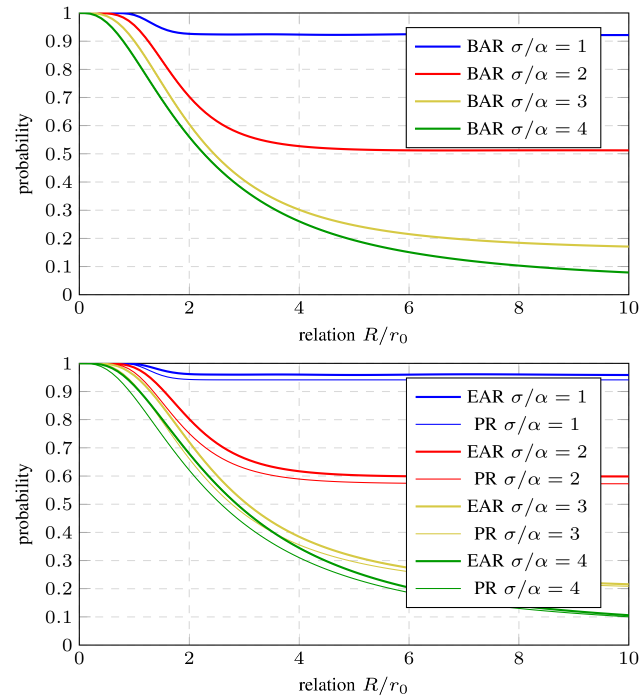


Fig. 9. Probability of basic acyclic (BAR), extended acyclic (EAR) and proper redundancy (PR) for different $\sigma/\alpha$ over $R/r_0$.

Adopting the previous notation the probability that two intersecting edges satisfy the extended acyclic redundancy is given by:

$$
\begin{aligned}
p_{ear} = & p_a(c)p_a(d) + p_a(e)p_a(f)(1 - p_a(c)p_a(d)) \\
& + p_b(c)p_b(f)(1 - p_a(d))(1 - p_a(e)) \\
& + p_b(d)p_b(e)(1 - p_a(c))(1 - p_a(f)) \\
& + p_a(c)p_a(e)(1 - p_a(d))(1 - p_a(f)) \\
& + p_a(d)p_a(f)(1 - p_a(c))(1 - p_a(e))
\end{aligned}
$$

The probability of a random intersection satisfying Euclidean extended acyclic redundancy is then given by

$$\int_0^{\infty}\int_0^{a}\int_0^{\pi}\int_0^{a}\int_0^{b} (p_{ear} f_{int})(a, b, x, y, \gamma)\mathrm{d}y\mathrm{d}x\mathrm{d}\gamma\mathrm{d}b\mathrm{d}a$$

## V. Numerical Evaluation

Based on the previous stochastic analysis we study numerically how far basic acyclic and extended acyclic redundancy can be expected to hold in graphs pertaining to the log-normal shadowing model. Moreover, based on the numerical analysis in [26] we study how likely these properties are compared to coexistence and proper redundancy.

As can be seen in equation (2) the probability that an edge of length $d$ exists under log-normal shadowing depends on the average communication distance $r_0$ and the relation between path loss coefficient $\alpha$ and log-normal shadowing variance $\sigma$. In particular, as shown in [26] the likelihood of coexistence and redundancy is independent of $r_0$ but depends only on the relation between $\alpha$ and $\sigma$. The same derivation holds for any thinning of the intersection space, especially for the probability expressions for basic and extended acyclic redundancy derived in this work.

Fig. 8 shows the likelihood of basic and extended acyclic redundancy and compares it to coexistence and proper redundancy. As can be seen in the range of $0 \leq \sigma/\alpha \leq 3$, the probability of any redundancy dominates that of coexistence. Moreover, the probability of extended acyclic redundancy dominates that of all other properties in that range. When $\sigma/\alpha$ is clear beyond 1 and all properties are unlikely, the curve progression of any redundancy falls marginally below coexistence.

Similar to [26] we study how far the properties are improved when artificially cutting the communication range to a maximum communication distance, i.e. edges of length greater than a certain value $R$ are ignored. We investigate different relations between log-normal variance $\sigma$ and path loss coefficient $\alpha$.

Fig. 9 shows the likelihood of basic acyclic redundancy and its extended variant depending on the relation between the artificially cut communication distance $R$ and the average communication distance $r_0$ of the log-normal shadowing model.

For all $\sigma/\alpha$ relations the probability tends to 1 when $R/r_0$ tends to zero. This is due to the fact that the smaller that relation the more the graph will be unit disk like which implies all considered properties.

As well, the smaller the relation $\sigma/\alpha$, the less will be the shadowing variation compared to the path loss coefficient and thus the graph faster tends towards a unit disk graph when the relation is decreased. Thus, for increasing $\sigma/\alpha$ the curves lie clearly below of each other.

On the other side, if the relation $R/r_0$ tends to infinity, the graph tends to the regular log-normal shadowing model. Thus, the probability values for acyclic redundancy (as well as proper redundancy and coexistence) tend to the ones under the regular log-normal shadowing model. As can be seen, all plotted curves tend to the probability values which can be determined from Fig. 8 by the basic and extended acyclic

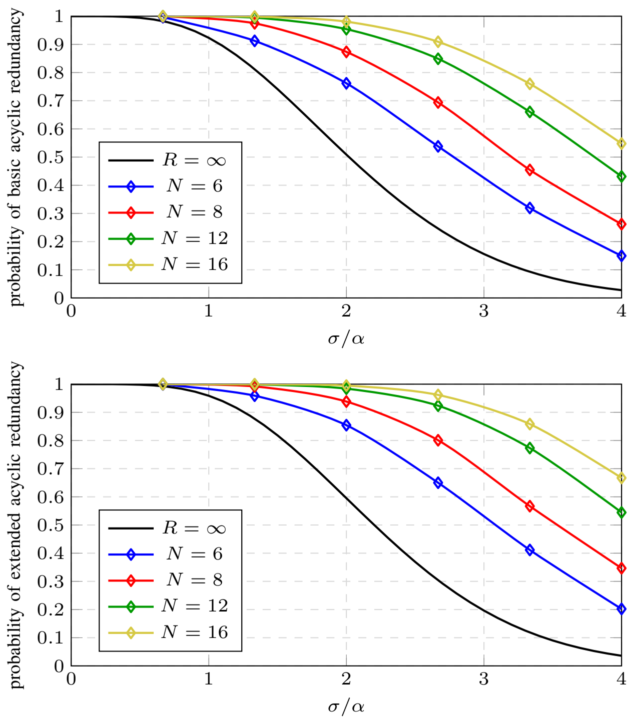


Fig. 10. Probability of basic acyclic and extended acyclic redundancy over $\sigma/\alpha$ at the empirical percolation radius $\bar{R}$ for varying average number $N$ of neighbors.

redundancy curves for $\sigma/\alpha = 1, 2, 3, 4$.

Basic acyclic redundancy compared to proper redundancy allows for two intersecting edges $uv$ and $wx$ less combinations for existence and non-existence of edges $uw$, $ux$, $vw$ and $vx$. It thus can be at most as likely as proper redundancy. However, the extended variant of acyclic redundancy allows use of the shorter intersecting edge to detour the longer one. Note that this edge exists anyway. As can be seen in the second plot of Fig. 9 the flexibility to use the fourth node to detour outweighs the constraints of basic acyclic redundancy posed on proper redundancy. Compared to proper redundancy, extended acyclic redundancy improves the likelihood for all relevant parameter settings.

For all properties we observe that probabilities tend to 1 when the relation $R/r_0$ and thus $R$ itself tends to 0. This is again because the graph gets the more unit disk like the more the artificial maximum communication distance $R$ is below average communication distance $r_0$. Such a graph always satisfies coexistence and any redundancy. However, the smaller one chooses $R$, the more the graph is partitioned. By setting $R = 0$ every vertex becomes isolated. Thus, as done in our precursor [26], to judge how far the probabilities can be improved by artificially setting a reasonable maximum communication distance we investigate the graphs at the percolation bounds regarding $R/r_0$.

Here, percolation bound refers to the minimum setting of $R$ to have a non-zero probability of finding an infinite connected component in random instances of graphs with infinite many nodes homogeneously Poisson distributed on an infinitely large plane. We call this $R$ the percolation radius.

Note that we use percolation bound and not connectivity probability. This keeps our measurement free of an additional parameter which would be the size of the observation window otherwise.

Following the same approach as in [26] we employ the simulation method from [29] to determine the percolation bounds. Percolation depends on the expected number of neighbors. Thus, the parameter depends on the density of the Poisson point process, as well as on the relation between $\sigma/\alpha$. We resort to the percolation radii we already determined empirically in [26] (see Fig. 9 in that paper). There we found $R$ depending on the PPP density $\lambda$ measured in average number $N$ of neighbor nodes under log-normal shadowing. We have determined $R$ exemplarily for $N = 6, 8, 12, 16$.

In Fig. 10 we show the probability of basic (the first plot) and extended acyclic redundancy (the second plot) for the empirically determined percolation radii $R$ for $N = 6, 8, 12, 16$. The black curves refer to the graph without artificially setting a maximum communication distance.

As can be seen in both plots, with more neighbors on average the probability of all properties improves. This, is obviously true since more neighbors on average supports smaller artificially set maximum communication distance $R$ which supports all redundancy property variants and coexistence as already discussed.

## VI. Conclusion

We investigated the likelihood of structural graph properties in random wireless communication graphs. Edges are sampled independent with a probability depending on the communication distance. In our numerical evaluations we have specifically looked at graphs where edge sampling follows the log-normal shadowing model.

Our work is a successor of our work [25] where we described a local algorithm for finding plane subgraphs under proper redundancy and coexistence. This work carries on our previous results, describing a novel local algorithm which requires only one property, acyclic redundancy.

We have seen that likelihood of basic acyclic redundancy is admittedly dominated by that of proper redundancy, however it makes the accessibility of vertices in the interior of triangles, resp. the coexistence property, irrelevant for the existence of a connectivity-preserving plane subgraph. However, extended acyclic redundancy is more likely than proper redundancy in relevant settings as well. Thus, it makes the new method more likely to succeed than the previous one for these two reasons.

The presented algorithms are simpler than the one presented in [25] as one has to consider just triangles and quadrilaterals instead of arbitrary large cliques. Moreover the new variants are more efficient, since each edge can be decided immediately once the 2-hop information are available and no waiting period is necessary.

In this paper BAR and EAR are formulated as local properties, i.e. it only depends on the existence of the edges within the four endpoints of an intersection. However, the properties could be transferred to a global version. So an intersection satisfies the global BAR if one of the intersecting edges is the longest edge of any triangle. Analogously the EAR can be globalized. Note that now one can allow not just detours of length two or three, i.e. an intersection satisfies the global ($k$)-EAR if one of the intersecting edges is the longest edge of a polygon of length at most $k$ (for an arbitrary large but fixed $k \geq 3$). These variants would remain edge-1-hop locally (global BAR) resp. $\lceil \frac{k}{2} \rceil$-hop locally (global EAR) detectable and the algorithms would still work (one has just to include the $k-1$-detours). However, now the decision if an intersection satisfies one of these properties depends on the existence of the edges to all other nodes (or at least in an large enough area in the Euclidean case). Thus, it would depend on infinitely many (resp. arbitrary many) edges. That makes the stochastical and numerical analysis much more difficult.

Our numerical evaluations show for which parameter settings we can assume the required properties to hold very likely, i.e., the parameter settings where it is reasonable to apply the studied local algorithm to construct plane subgraphs. An analysis of the likelihood that such algorithm variant fails is subject to future work. Moreover, our stochastic analysis strongly depends on the Poisson point process assumption. In future, other commonly used point process models can be investigated. Furthermore an investigation of different methods to evaluate/rank the edges to obtain several total orders are an interesting field of study.

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