Downlink Analysis for a Heterogeneous Cellular Network

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***Abstract*—In this paper, a comprehensive study of the downlink performance in a heterogeneous cellular network (or hetnet) is conducted. A general hetnet model is considered consisting of an arbitrary number of open-access and closed-access tier of base stations (BSs) arranged according to independent homogeneous Poisson point processes. The BSs of each tier have a constant transmission power, random fading coefficient with an arbitrary distribution and arbitrary path-loss exponent of the power-law path-loss model. For such a system, analytical characterizations for the coverage probability and average rate at an arbitrary mobile-station (MS), and average per-tier load are derived for the max-SINR connectivity model. Using stochastic ordering, interesting properties and simplifications for the hetnet downlink performance are derived by relating this connectivity model to the maximum instantaneous received power (MIRP) connectivity model, providing good insights about the hetnets and the downlink performance in these complex networks. Furthermore, the results also demonstrate the effectiveness and analytical tractability of the stochastic geometric approach to study the hetnet performance.**

***Index Terms*—Multi-tier networks, Cellular Radio, Co-channel Interference, Fading channels, Poisson point process, max-SINR connectivity.**

I. Introduction

THE modern cellular communication network is an overlay of multiple contributing subnetworks such as the macrocell, microcell, picocell and femtocell networks, collectively called the heterogeneous network (or, in short, *hetnets*). The hetnets have been shown to sustain greater end-user data-rates and throughput as well as provide indoor and cell-edge coverage, further leading to their inclusion as an important feature to be implemented under the fourth-generation (4G) cellular standards [4]–[10].

Until recently, the analysis of such networks has been done solely through system simulations. Since the hetnets consist of a combination of regularly spaced macrocell base-stations (BSs) along with irregularly spaced microcell and picocell BSs and often randomly placed end-user deployed femtocell BSs, it is difficult to study the entire network at once using simulations. Further, the BSs in each of these networks have different transmission powers, traffic-load carrying capabilities and different radio environment that is based on the locations in which they are deployed. The many parameters involved in the design and modeling of the individual networks makes it difficult to narrow all the possibilities down to a limited set of simulation scenarios based on which one can make design decisions for the entire network. Under these circumstances, the development of an analytical model that captures all the design scenarios of interest is of great importance.

The hetnet performance is studied by viewing the hetnet as composed of multiple tiers of networks (e.g. macrocell, microcell, picocell and femtocell networks), each modeled as an independent homogeneous Poisson point process, and such studies have been done in [11]–[18] and by us in [1]–[3]. These studies mathematically characterize important performance metrics such as coverage probability (1 - outage probability), average ergodic rate, average load carried by BSs of each tier and load-awareness. Furthermore, such studies have facilitated the characterization of the improvements that techniques such as fractional frequency reuse and carrier aggregation bring to cellular performance as well as hetnet performance. In the following subsection, we differentiate our work from the other prior work on hetnets and list the contributions of this paper.

Contributions of the paper

Here, the hetnet is modeled to consist of open and closed access networks formed by the arrangement of BSs according to homogeneous Poisson point process with a certain density for each tier, and independent of the other tiers. The focus is on the downlink performance analysis; the MS has access to only the open-access tiers and connects to one of the BSs in these tiers. The closed access tiers only cause interference at the MS. Hence, we study the downlink performance where the hetnet consists of an arbitrary number of open and closed access tiers. Signals from BSs of a given tier have a constant transmit power, random fading coefficient that is i.i.d. across all the BSs of the same tier and independent of those of the other tiers with any arbitrary distribution, arbitrary path-loss exponent that is constant for all BSs of the same tier and different across different tiers, and the signal-to-interference-plus-noise-ratio (SINR) threshold for connectivity to a given k^{th} open-access tier's BS is $\beta_k,\ k = 1, \cdots, K$. For such a general setting, the coverage probability at the MS is derived for the max-SINR connectivity model. In the max-SINR connectivity model, the MS is said to be in coverage if there exists at least one open-access BS with an SINR above the corresponding threshold.

The results shown here are generalizations of the existing results in [2], [3], [12], [16], [18]. In [14], [15], the coverage probability results are obtained for the hetnets under the

A special cases of the results in this paper were presented in [1]–[3]

max-SINR connectivity, but for the case where the fading coefficients for the BS transmissions are independent and identically distributed (i.i.d.) exponential random variables, and the path-loss exponents are the same for all the tiers. Using an entirely different approach, [11]–[13] derives the hetnet coverage probability but are again restricted to the i.i.d. exponential distribution case for fading. In [1]–[3], we derived the hetnet coverage probability for the case when the i.i.d. fading coefficients have an arbitrary distribution and the path-loss exponents are different for different tiers, for the maximum instantaneous received power (MIRP) connectivity model, which is a special case for the max-SINR connectivity model, as will be discussed later. Here, we derive the coverage probabilities for the general connectivity model (max-SINR) for the general system settings mentioned above.

When the SINR thresholds of all the tiers are above 1, the hetnet coverage probability under max-SINR connectivity and MIRP connectivity are identical. Further, in these special cases, simple analytical expression are derived for the coverage probability, average rate and the load carried by the BSs of each tier. The following section describes the system model in detail.

II. System Model

This section describes the various elements used to model the wireless network, namely, the BS layout, the radio environment, and the performance metrics of interest.

1) BS Layout: The hetnet is composed of K open-access tiers and L closed-access tier, and the BS layout in each tier is according to an independent homogeneous Poisson point process in $\mathbb{R}^2$ with density λ_{ok}, λ_{cl} for the k^{th} open-access tier and l^{th} closed-access tier, respectively, where $k = 1, \ldots,\ K$ and $l = 1, \ldots,\ L$. The MS is allowed to communicate with any BS of the open-access tiers, but cannot communicate with any of the closed-access BSs.

2) Radio Environment and downlink SINR: The signal transmitted from each BS undergoes shadow fading and path-loss. The SINR at an arbitrary MS in the system from the i^{th} BS of the k^{th} open-access tier is the ratio of the received power from this BS to the sum of the interferences from all the other BSs in the system and the constant background noise η, and is expressed as

$$\mathrm{SINR}_{ki} = \frac{P_{ok}\Psi_{ki}R_{ki}^{-\varepsilon_k}}{I_o - P_{ok}\Psi_{ki}R_{ki}^{-\varepsilon_k} + I_c + \eta}, \tag{1}$$

where $I_o = \sum_{m=1}^{K}\sum_{n=1}^{\infty} P_{om}\Psi_{mn}R_{mn}^{-\varepsilon_m}$ is the sum of the received powers from all the open-access tier BSs, $\{P_{om}, \Psi_{mn}, \varepsilon_m, R_{mn}\}_{m=1,\ n=1}^{m=K,\ n=\infty}$ are the constant transmit power, random shadow fading factor, constant path-loss exponent and the distance from the MS of the n^{th} BS of the m^{th} open-access tier. Similarly, $I_c = \sum_{l=1}^{L}\sum_{n=1}^{\infty} P_{cl}\Psi_{cln}R_{cln}^{-\varepsilon_{cl}}$ is the sum of the received powers from all the closed-access tier BSs, $\{P_{cl}, \Psi_{cln}, \varepsilon_{cl}, R_{cln}\}_{l=1,\ n=1}^{l=L,\ n=\infty}$ lists the constant transmit power, random shadow fading factor, and the constant path-loss exponent of the n^{th} BS of the l^{th} closed-access tier. The fading coefficients $\{\Psi_{mn}\}_{n=1}^{\infty}$ ($\{\Psi_{cln}\}_{n=1}^{\infty}$) are i.i.d. random variables with the same distribution as Ψ_m (Ψ_{cl}), $m = 1, \ldots,\ K$ ($l = 1, \ldots,\ L$). Further, following [19], it is assumed that $\left\{\mathbb{E}\left[\Psi_m^{\frac{2}{\varepsilon_m}}\right]\right\}_{m=1}^{K}$, $\left\{\mathbb{E}\left[\Psi_{cl}^{\frac{2}{\varepsilon_{cl}}}\right]\right\}_{l=1}^{L} < \infty$. Finally, R_{mn} (R_{cln}) is the distance of the n^{th} nearest BS belonging to the m^{th} open-access (l^{th} closed-access) tier, and $\{R_{mn}\}_{n=1}^{\infty}$, $\{R_{cln}\}_{n=1}^{\infty}$ represents the distance from origin of the sets of points distributed according to the homogeneous Poisson point processes described in Section II-1. The various symbols introduced in this section are listed in Table I for quick reference.

3) BS connectivity models: A MS is able to communicate with a BS of the k^{th} open-access tier if the corresponding SINR is above a certain threshold $\beta_k,\ k = 1, \cdots,\ K$. In this case, the MS is said to be in coverage. The BS connectivity models provide a rule to determine which BS to connect to, and in this paper, we focus on the max-SINR connectivity model. The MIRP connectivity model is a special case of the max-SINR connectivity model, and will be discussed in detail in the later sections.

Under the max-SINR connectivity model, the MS is said to be in coverage if there exists at-least one BS among all the open-access tiers with an SINR at the MS above the corresponding threshold, and is mathematically expressed as follows.

$$\begin{aligned}\mathbb{P}_{\text{coverage}}^{\max-\text{SINR}} &= \mathbb{P}\left(\bigcup_{k=1}^{K}\bigcup_{i=1}^{\infty}\{\mathrm{SINR}_{ki} > \beta_k\}\right)\\ &= \mathbb{P}\left(\bigcup_{k=1}^{K}\{\mathrm{SINR}_k\,(\max) > \beta_k\}\right),\end{aligned} \tag{2}$$

where SINR_{ki} corresponds to the i^{th} BS of the k^{th} tier as defined in (1) and $\mathrm{SINR}_k\,(\max)$ is the maximum SINR at the MS among all the k^{th} open-access tier BSs.

In the following section, we derive expressions for the hetnet coverage probability for the above mentioned connectivity models.

III. Hetnet Coverage Probability

In [20], a technique to compute the downlink coverage probability under max-SINR connectivity for a single-tier network was shown. In [15], this technique is used to compute the hetnet coverage probability for an open-access case where the fading coefficients for all the BSs in the system are i.i.d. unit mean exponential random variables and the path-loss exponents are the same for all tiers. Here, we generalize the technique developed in [20] to compute the hetnet coverage probability for both the max-SINR and nearest-BS connectivity models for a general system model explained in Section II.

The coverage probability expressions in (2) can be equivalently expressed as follows:

$$\begin{aligned}\mathbb{P}_{\text{coverage}}^{\max-\text{SINR}} &= \mathbb{P}\left(\bigcup_{k=1}^{K}\left\{\frac{M_k}{I_o + I_c + \eta - M_k} > \beta_k\right\}\right)\\ &= \mathbb{P}\left(\left\{\max_{k=1,\cdots,\ K}\gamma_k M_k > I_o + I_c + \eta\right\}\right),\end{aligned} \tag{3}$$

Symbol	Description
$K,\ L$	Number of open-access and closed-access tiers, respectively.
$\{\lambda_{ok}\}_{k=1}^{K},\ \{\lambda_{cl}\}_{l=1}^{L}$	BS densities of open-access and closed-access tiers, respectively.
$\{P_{ok}\}_{k=1}^{K},\ \{P_{cl}\}_{l=1}^{L}$	Constant transmission powers of the BSs of the K open-access tiers and closed access tier, respectively
$\{\varepsilon_k\}_{k=1}^{K},\ \{\varepsilon_{cl}\}_{l=1}^{L}$	Path-loss exponents of the open and closed - access tiers (> 2).
$\{\Psi_k\}_{k=1}^{K},\{\Psi_{cl}\}_{l=1}^{L}$	i.i.d. fading gains of the open and closed-access tiers $\left(\mathbb{E}\Psi_k^{\frac{2}{\varepsilon_l}},\ \mathbb{E}\Psi_{cl}^{\frac{2}{\varepsilon_{cl}}}<\infty\right)$
$\{\beta_k\}_{k=1}^{K}$	SINR thresholds for connectivity to a BS in the k^{th} open-access tier
η	Background noise power
$\{\gamma_k\}_{k=1}^{K}$	$=\left\{1+\frac{1}{\beta_k}\right\}_{k=1}^{K}$

TABLE I
LIST OF SYMBOLS USED IN THE PAPER

where $M_k = \max_{n=1,\cdots,\infty} P_{ok}\Psi_{okl}R_{kl}^{-\varepsilon_k}$ is the maximum of the received powers from all the k^{th} tier BSs, I_o (I_c) is the sum of the received powers from all the open-access BSs (closed-access BSs) in the system, and are defined in (1). We begin with computing the Laplace transform of the interference from the closed-access tiers, I_c, $\mathcal{L}_{I_c}(s) = \mathbb{E}\left[\mathrm{e}^{-sI_c}\right]$.

Lemma 1. *The Laplace transform of the interference from the closed-access tiers is*

$$\mathcal{L}_{I_c}(s) \quad = \quad \mathrm{e}^{-\sum_{l=1}^{L}\lambda_{cl}\pi(sP_{cl})^{\frac{2}{\varepsilon_{cl}}}\mathbb{E}\left[\Psi_{cl}^{\frac{2}{\varepsilon_{cl}}}\right]\Gamma\left(1-\frac{2}{\varepsilon_{cl}}\right)}. \quad (4)$$

Proof: The proof for (4) is as follows. $\mathcal{L}_{I_c}(s) = \mathbb{E}\left[\exp\left(-s\sum_{l=1}^{L}\sum_{n=1}^{\infty}P_{cl}\Psi_{cln}R_{cln}^{-\varepsilon_{cl}}\right)\right] \overset{(a)}{=} \prod_{l=1}^{L}\mathbb{E}\left[\exp\left(-s\sum_{n=1}^{\infty}P_{cl}\Psi_{cln}R_{cln}^{-\varepsilon_{cl}}\right)\right] \overset{(b)}{=} \prod_{l=1}^{L}\exp\left(-\lambda_{cl}\mathbb{E}_{\Psi_{cl}}\left[\int_{r=0}^{\infty}\left(1-\mathrm{e}^{-sP_{cl}\Psi_{cl}r^{-\varepsilon_{cl}}}\right)2\pi r dr\right]\right)$, where (a) is obtained because the BS arrangement for the L closed-access tiers and the corresponding transmission and fading parameters are independent of each other, and (b) evaluates the expectation in (a) using the Campbell's theorem of Poisson point process [21, Page 28], and (4) is obtained by evaluating the integral in (b). ■

Next, we derive the expression for the Laplace transform that will be used to obtain semi-analytical expressions for $\mathbb{P}_{\text{coverage}}^{\max-\text{SINR}}$.

Lemma 2.

$$\begin{aligned}&\mathcal{L}_{I_o+I_c+\eta,\ \max_{k=1,\cdots,K}\gamma_k M_k\le u}(s)\\&\triangleq \mathbb{E}\left[\mathrm{e}^{-s(I_o+I_c+\eta)}\mathcal{I}\left(\max_{k=1,\cdots,K}\gamma_k M_k\le u\right)\right]\\&=\mathcal{L}_{I_c}(s)\exp\left(-s\eta-\sum_{k=1}^{K}\lambda_{ok}\pi\left(sP_{ok}\right)^{\frac{2}{\varepsilon_k}}\mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right]\times\right.\\&\left.\left[\Gamma\left(1-\frac{2}{\varepsilon_k}\right)+\frac{2}{\varepsilon_k}\Gamma\left(-\frac{2}{\varepsilon_k},\frac{su}{\gamma_k}\right)\right]\right),\end{aligned} \quad (5)$$

where $\mathcal{L}_{I_c}(s)$ is from Lemma 1 and the random variables Ψ_{k1} and Ψ_k are i.i.d. for all $k=1,\cdots,K$.

Proof: Please refer [22, Appendix A] for the proof. ■

The significance of Lemmas 1 and 2 are as follows. Notice from (3) that the hetnet coverage probability can be obtained if the joint probability density function (p.d.f.) of $\left(I_o+I_c+\eta,\ \max_{i=1,\cdots,K}\gamma_i M_i\right)$ is known. The joint p.d.f.s can be derived from the Laplace transform expressions in Lemma 2 using the following simple operations.

$$\begin{aligned}&f_{I_o+I_c+\eta,\max_{i=1,\cdots,K}\gamma_i M_i}(x,y) =\\&\int_{\omega=-\infty}^{\infty}\frac{\partial}{\partial u}\mathcal{L}_{I_o+I_c+\eta,\ \max_{i=1,\cdots,K}\gamma_i M_i\le u}(j\omega)\bigg|_{u=y}\frac{\mathrm{e}^{j\omega x}}{2\pi}d\omega.\end{aligned} \quad (6)$$

where $f_{\cdot,\cdot}(\cdot,\cdot)$ denotes the joint p.d.f. of the involved random variables. This is shown for the max-SINR connectivity case in [20, Corollary 4]. Further, the partial derivative term in the above equation can be easily computed and is given below.

$$\begin{aligned}&\frac{\frac{\partial}{\partial u}\mathcal{L}_{I_o+I_c+\eta,\ \max_{i=1,\cdots,K}\gamma_i M_i\le u}(s)}{\mathcal{L}_{I_o+I_c+\eta,\ \max_{i=1,\cdots,K}\gamma_i M_i\le u}(s)} =\\&\sum_{k=1}^{K}\lambda_k\frac{2\pi}{\varepsilon_k}\left(\gamma_k P_k\right)^{\frac{2}{\varepsilon_k}}\mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right]u^{-1-\frac{2}{\varepsilon_k}}\mathrm{e}^{-\frac{su}{\gamma_k}}.\end{aligned} \quad (7)$$

When fading coefficients are i.i.d. unit mean exponential random variables $\mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right]=\Gamma\left(1+\frac{2}{\varepsilon_k}\right)$, setting $\{\lambda_{cl}\}_{l=1}^{L}=0$ and $\{\varepsilon_k\}_{k=1}^{K}=\alpha$, (7) reduces to [15, (2)]. Having computed the expressions for the joint p.d.f.'s in (6), the coverage probability can be easily obtained as shown below.

Theorem 1. *The hetnet coverage probability under max-SINR connectivity model is as follows:*

$$\begin{aligned}&\mathbb{P}_{\text{coverage}}^{\max-\text{SINR}} = \sum_{i=1}^{K}\lambda_i\frac{2\pi}{\varepsilon_i}\left(\gamma_i P_i\right)^{\frac{2}{\varepsilon_i}}\mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right]\times\\&\int_{y=0}^{\infty}\int_{\omega=-\infty}^{\infty}\mathcal{L}_{I_o+I_c+\eta,\ \max_{i=1,\cdots,K}\gamma_i M_i\le y}(j\omega)\times\\&\frac{\left(\mathrm{e}^{j\omega y\left(1-\gamma_i^{-1}\right)}-\mathrm{e}^{j\omega\left(\eta+y\left(\kappa^{-1}-\gamma_i^{-1}\right)\right)}\right)}{2\pi j\omega y^{1+\frac{2}{\varepsilon_i}}}d\omega dy,\end{aligned} \quad (8)$$

where $\kappa = \max_{i=1,\cdots,K}\gamma_i$, all the other symbols are in Table I, and the Laplace transform function in (8) is given in (5).

Proof: Please refer [22, Theorem 1] for the proof. ■

Using an alternate approach, expressions for the hetnet coverage probability are obtained in [11], again, when all the fading coefficients are i.i.d. exponential random variables.

For a general system model as in this paper, to the best of our knowledge, the hetnet coverage probability has not been characterized, until now.

Nevertheless, the semi-analytical expressions are extremely complicated even for numerical computations, and little intuition and insights about the hetnet performances are obtainable from these expressions. As a result, a more qualitative study is imperative to better understand these soon-to-be-prevalent cellular networks. From now onwards, we conduct a more systematic study to bring out the properties and dependencies of the hetnet performance on the various parameters of the system. To begin with, we make the following observations about the hetnet performance.

Corollary 1. *The downlink coverage probability in the hetnet is the same as in another hetnet with the same open-access tiers as in the original hetnet (described in Section II-1) and one closed access tier where the BSs have unity transmission power, fading coefficient and path-loss exponent and are arranged according to a non-homogeneous Poisson point process with a BS density function*

$$\lambda_c(r) \;=\; \sum_{l=1}^{L} \lambda_{cl} P_{cl}^{\frac{2}{\varepsilon_{cl}}} \mathbb{E}\left[\Psi_{cl}^{\frac{2}{\varepsilon_{cl}}}\right] r^{\frac{2}{\varepsilon_{cl}}-1},\; r \geq 0. \quad (9)$$

Proof: Please refer [22, Corollary 1] for the complete proof. ∎

Hence, we have shown an equivalence between a hetnet with L closed-access tiers and another hetnet with a single closed-access tier. Next, we make an interesting observation regarding the hetnet downlink performance under the max-SINR connectivity model.

Corollary 2. *The hetnet performance under max-SINR connectivity with an arbitrary fading distribution at each tier is the same as in another hetnet with open-access and closed-access BS densities as* $\left\{\lambda_{oi}\mathbb{E}\Psi_{oi}^{\frac{2}{\varepsilon_i}} \Big/ \Gamma\left(1+\frac{2}{\varepsilon_i}\right)\right\}_{i=1}^{K}$ *and* $\left\{\lambda_{ci}\mathbb{E}\Psi_{ci}^{\frac{2}{\varepsilon_{ci}}} \Big/ \Gamma\left(1+\frac{2}{\varepsilon_{ci}}\right)\right\}_{i=1}^{L}$*, respectively, and i.i.d. unit mean exponential distribution for fading at all the BSs in the network.*

The above result is obtained by noting that the effect of fading is equivalent to scaling the density of BSs by the $\frac{2}{\varepsilon}^{\text{th}}$ moment of the fading random variable, due to [19, Corollary 2]. A large body of work involving the stochastic geometric study of networks predominantly assume fading coefficients to be i.i.d. exponential random variables, as this greatly simplifies the analysis and renders itself to closed-form characterization of coverage probabilities and other related performance metrics of several networks including the hetnets (see [14]). A common criticism for all these works has been that the exponential distribution does not accurately capture the slow fading environment. Interestingly, the above corollary shows an example of a scenario wherein studies with exponential fading assumptions completely characterizes the arbitrary fading scenario.

The importance of the Corollary 1 is that the SINR distribution of the two equivalent hetnets are the same. Hence, without loss of generality, we study the downlink performance where the hetnet consists of K tiers of open access networks and a single closed access network. For the sake of simplicity, it is assumed that the closed-access tier has homogeneous Poisson point process based BS arrangement with a constant BS density λ_c, transmission power P_c, path-loss exponent ε_c and i.i.d. fading coefficients with the same distribution as Ψ_c $\left(\mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon_c}}\right] < \infty\right)$.

IV. Qualitative study of hetnet downlink performance

When $\{\beta_k\}_{k=1}^{\infty} = \beta$, commonly referred to as the unbiased case in the literature, the hetnet coverage probabilities of the max-SINR connectivity model is identical to the maximum instantaneous received power (MIRP) connectivity model. Under the MIRP connectivity, the MS connects to the BS with the maximum instantaneous received power among all the open-access tiers. As a result, the serving BS and the coverage probability expression for the MIRP are

$$\begin{aligned} (T, I) &= \underset{k=1,\cdots,\ K,\ i=1,\ 2,\cdots}{\operatorname{argmax}} P_{ok}\Psi_{ki}R_{ki}^{-\varepsilon_k}, \\ \mathbb{P}_{\text{coverage}}^{\text{MIRP}} &= \mathbb{P}\left(\{\text{SINR}_{T,I} > \beta_T\}\right), \end{aligned} \quad (10)$$

where T refers to the tier-index and I refers to the BS-index of the serving BS. Next, we characterize the c.c.d.f. of $\text{SINR}_{T,I}$ for the MIRP case, and several related important characteristics.

A. SINR characterization under MIRP connectivity

The following stochastic equivalence helps simplify the SINR characterization.

Lemma 3. *The SINR at the MS under MIRP is the same as in the two-tier hetnet where the tier to which the MS has an open-access network with a BS density function* $\tilde{\lambda}(r) = \sum_{k=1}^{K}\tilde{\lambda}_k(r)$ *with* $\tilde{\lambda}_k(r) = \lambda_k \frac{2\pi}{\varepsilon_k} P_k^{\frac{2}{\varepsilon_k}} \mathbb{E}\Psi_k^{\frac{2}{\varepsilon_k}} r^{\frac{2}{\varepsilon_k}-1},\ r \geq 0$ *and a closed-access network with a BS density function* $\hat{\lambda}(r) = \lambda_c \frac{2\pi}{\varepsilon_c} P_c^{\frac{2}{\varepsilon_c}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon_c}}\right] r^{\frac{2}{\varepsilon_c}-1}$*. All the BSs in the equivalent systems have unity transmit powers, fading coefficients and path-loss exponents. The SINR is stochastically equal to*

$$\begin{aligned} &\text{SINR}_{T,I} \\ &=_{\text{st}} \left.\frac{\tilde{R}_{T,1}^{-1}}{\sum_{\substack{k=1 \\ (k,l)\neq(T,1)}}^{K}\sum_{l=1}^{\infty}\tilde{R}_{kl}^{-1} + \sum_{l=1}^{\infty}\hat{R}_l^{-1} + \eta}\right|_{\left(\{\tilde{\lambda}_k(r)\}_{k=1}^{K},\hat{\lambda}(r)\right)} \\ &=_{\text{st}} \left.\frac{\tilde{R}_1^{-1}}{\sum_{k=2}^{\infty}\tilde{R}_k^{-1} + \sum_{l=1}^{\infty}\hat{R}_l^{-1} + \eta}\right|_{\left(\tilde{\lambda}(r),\hat{\lambda}(r)\right)}, \end{aligned} \quad (11)$$

where $=_{\text{st}}$ *indicates the equivalence in distribution; and* $\left\{\tilde{R}_i\right\}_{i=1}^{\infty}$ $\left(\left\{\hat{R}_i\right\}_{i=1}^{\infty}\right)$ *is the ascendingly ordered distances of the BSs from the origin, obtained from a non-homogeneous 1-D Poisson point process with BS density function* $\tilde{\lambda}(r)$ $\left(\hat{\lambda}(r)\right)$ *defined above.*

Proof: Please refer [22, Lemma 3] for the complete proof. ∎

The following lemma shows interesting stochastic equivalences when $\{\varepsilon_k\}_{k=1}^K = \varepsilon_c = \varepsilon$.

Lemma 4. *The hetnet SINR under MIRP connectivity has the same distribution as in the following three networks. The first is a hetnet with BS densities* $\left\{\lambda_k P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon}}\right]\right\}_{k=1}^K$, $\lambda_c P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right]$ *for the* K *open-access tiers and the closed-access tier, respectively, unity transmit powers and shadow fading factors for all tiers. The other two are two-tier networks with unity transmit powers and shadow fading factors for all their BSs. The first two-tier network has the open-access tier BS density* $\sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right]$, *closed-access tier BS density* $\lambda_c P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right]$ *and experiences the same background noise as the hetnets. The second two-tier network has a unity open-access tier BS density, closed-access tier BS density* $\hat{\lambda}_c = \frac{\lambda_c P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right]}{\sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right]}$ *and a background noise* $\bar{\eta} = \eta\left(\sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right]\right)^{-\frac{\varepsilon}{2}}$. *Equivalently,*

$$\mathrm{SINR}_{T,I} =_{\mathrm{st}} \mathrm{SINR}\left(K+1, \left\{\lambda_k P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon}}\right]\right\}_{k=1}^K, \lambda_c P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right], \eta, T\right) \tag{12}$$

$$=_{\mathrm{st}} \mathrm{SINR}\left(2, \sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right], \lambda_c P_c^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right], \eta, 1\right) \tag{13}$$

$$=_{\mathrm{st}} \mathrm{SINR}\left(2, 1, \hat{\lambda}_c, \bar{\eta}, 1\right), \tag{14}$$

where $=_{\mathrm{st}}$ *indicates equivalence in distribution. The SINR expression on the right-hand side is a function of the total number of tiers in the hetnet, BS densities of each open-access tier, BS densities of the closed-access tier, back-ground noise power, and the tier index of the serving BS, respectively.*

Proof: Please refer [22, Lemma 4] for the complete proof. ∎

Lemmas 3 and 4 are generalizations of [2, Lemma 1] and [23, Lemma 1], respectively, to the case where the hetnet also contains a closed-access tier. Next, we compute the hetnet coverage probability.

Theorem 2. *The hetnet coverage probability under MIRP is*

$$\mathbb{P}_{\mathrm{coverage}}^{\mathrm{MIRP}} = \sum_{k=1}^K \lambda_k P_k^{\frac{2}{\varepsilon_k}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right] \times \int_{r=0}^\infty 2\pi r \int_{\omega=-\infty}^\infty \frac{\mathrm{e}^{j\omega\eta r^{\varepsilon_k}}\left(1-\mathrm{e}^{-\frac{j\omega}{\beta_k}}\right)}{j\omega 2\pi} \times \mathrm{e}^{-\lambda_c P_c^{\frac{2}{\varepsilon_c}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon_c}}\right]\pi r^{\frac{2\varepsilon_k}{\varepsilon_c}} G\left(j\omega, \frac{2}{\varepsilon_c}\right)} \times \mathrm{e}^{-\sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon_l}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon_l}}\right]\pi r^{\frac{2\varepsilon_k}{\varepsilon_l}} {}_1F_1\left(-\frac{2}{\varepsilon_l};1-\frac{2}{\varepsilon_l};j\omega\right)} d\omega dr, \tag{15}$$

where $G\left(j\omega, \frac{2}{\varepsilon_c}\right) = \int_{t=0}^\infty \left(1-\mathrm{e}^{j\omega t}\right) \frac{2}{\varepsilon_c} t^{-1-\frac{2}{\varepsilon_c}} dt$.

Proof: The proof is along the same lines as [2, Theorem 1], and is not shown here. ∎

The above expression can be greatly simplified under certain special cases, and the following results present these cases.

Corollary 3. *When* $\{\varepsilon_k\}_{k=1}^K = \varepsilon_c = \varepsilon$, *the hetnet coverage probability is*

$$\mathbb{P}_{\mathrm{coverage}}^{\mathrm{MIRP}} = \sum_{k=1}^K \frac{\lambda_k P_k^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon}}\right] \int_{\omega=-\infty}^\infty \frac{\left(1-\mathrm{e}^{-\frac{j\omega}{\beta_k}}\right)}{j\omega 2\pi} H(j\omega)\, d\omega}{\sum_{l=1}^K \lambda_l P_l^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right]}, \tag{16}$$

$$H(j\omega) = \int_{r=0}^\infty 2\pi r \times \mathrm{e}^{j\omega\bar{\eta}r^\varepsilon - \pi r^2\left({}_1F_1\left(-\frac{2}{\varepsilon};1-\frac{2}{\varepsilon};j\omega\right)+\hat{\lambda}_c G\left(j\omega,\frac{2}{\varepsilon}\right)\right)} dr \tag{17}$$

where $H(j\omega)|_{\bar{\eta}=0} = \frac{1}{{}_1F_1\left(-\frac{2}{\varepsilon};1-\frac{2}{\varepsilon};j\omega\right)+\hat{\lambda}_c G\left(j\omega,\frac{2}{\varepsilon}\right)}$, $\bar{\eta}$ *and* $\hat{\lambda}_c$ *are from Lemma 4 and* $G(\cdot,\cdot)$ *is defined in Theorem 2. When* $\{\beta_k\}_{k=1}^K = \beta$ *or* $\{\beta_k\}_{k=1}^K \geq 1$, *(16) is equal to* $\mathbb{P}_{\mathrm{coverage}}^{\mathrm{max-SINR}}$. *When there is no closed-access tier* $\left(\hat{\lambda}_c = 0\right)$, *(16) is equal to the single-tier network coverage probability (see [19, Corollary 4]) and is independent of the transmission powers and fading factors of the BSs in the system.*

Proof: The result is obtained by exchanging the order of integrations in (15) and simplifying. ∎

The following theorem shows another scenario when the hetnet coverage probabilities are identical for the max-SINR and MIRP connectivity models.

Theorem 3. *When* $\beta_k \geq 1$, $\forall\ k = 1, \cdots, K$, *the hetnet coverage probability is given by*

$$\mathbb{P}_{\mathrm{coverage}}^{\mathrm{max-SINR}} = \mathbb{P}_{\mathrm{coverage}}^{\mathrm{MIRP}} = \sum_{k=1}^K \frac{\lambda_{ok} P_{ok}^{\frac{2}{\varepsilon_k}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon_k}}\right] \beta_k^{-\varepsilon_k}}{\Gamma\left(1+\frac{2}{\varepsilon_k}\right)} \times \int_{r=0}^\infty 2\pi r \times \mathrm{e}^{-\eta r^{\varepsilon_k} - \frac{\lambda_c \pi P_c^{\frac{2}{\varepsilon_c}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon_c}}\right] r^{\frac{2\varepsilon_k}{\varepsilon_c}}}{\Gamma\left(1+\frac{2}{\varepsilon_c}\right)\mathrm{sinc}\left(\frac{2\pi}{\varepsilon_c}\right)}} \mathrm{e}^{-\sum_{l=1}^K \frac{\lambda_{ol}\pi P_{ol}^{\frac{2}{\varepsilon_l}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon_l}}\right] r^{\frac{2\varepsilon_k}{\varepsilon_l}}}{\Gamma\left(1+\frac{2}{\varepsilon_l}\right)\mathrm{sinc}\left(\frac{2\pi}{\varepsilon_l}\right)}} dr, \tag{18}$$

and in the interference limited case $(\eta = 0)$ *when* $\{\varepsilon_k\}_{k=1}^K = \varepsilon_c = \varepsilon$

$$\mathbb{P}_{\mathrm{coverage}}^{\mathrm{max-SINR}} = \mathbb{P}_{\mathrm{coverage}}^{\mathrm{MIRP}} = \sum_{k=1}^K \frac{\lambda_{ok} P_{ok}^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_k^{\frac{2}{\varepsilon}}\right] \mathrm{sinc}\left(\frac{2\pi}{\varepsilon}\right) \beta_k^{-\varepsilon}}{\lambda_c P_c^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_c^{\frac{2}{\varepsilon}}\right] + \sum_{l=1}^K \lambda_{ol} P_{ol}^{\frac{2}{\varepsilon}} \mathbb{E}\left[\Psi_l^{\frac{2}{\varepsilon}}\right]}. \tag{19}$$

Proof: Firstly, from [14, Lemma 1], when $\beta_k \geq 1$, there exists at most one open-access BS that can have an SINR above the corresponding threshold. As a result, hetnet coverage probability in (2) becomes $\mathbb{P}_{\mathrm{coverage}}^{\mathrm{max-SINR}} = \sum_{k=1}^K \mathbb{P}\left(\{\mathrm{SINR}_k(\max) > \beta_k\}\right) = \mathbb{P}_{\mathrm{coverage}}^{\mathrm{MIRP}}$. See [22, Appendix D] to derive (18), which simplifies to (19) when $\eta = 0$. ∎

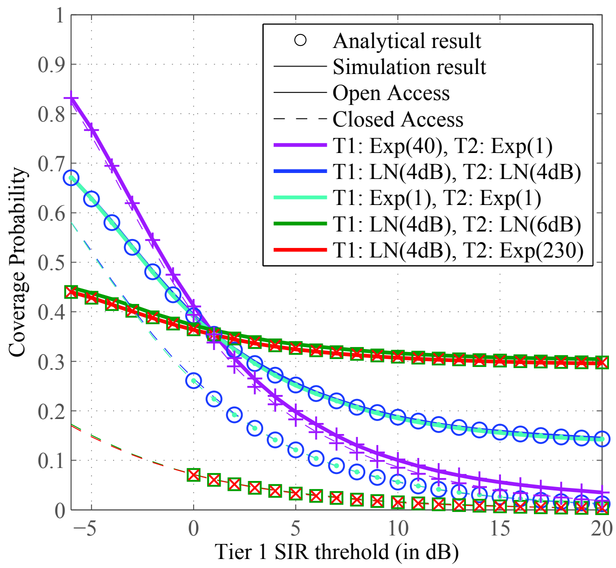

Fig. 1. Two-tier hetnet: Comparing coverage probabilities for various shadow fading distributions

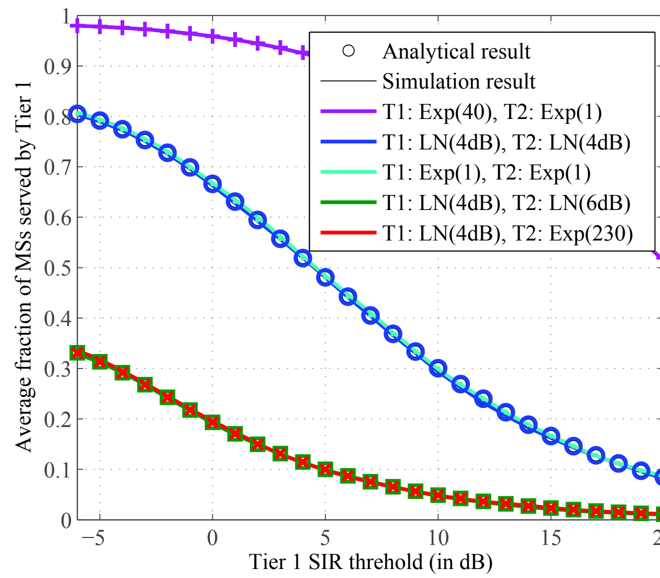

Fig. 2. Two-tier hetnet: Average fraction of MSs served by macrocell BSs vs macrocell SIR threshold

In the above result, (18) can be easily computed numerically and is an extension of [14, Theorem 1] to arbitrary fading and path-loss case. The study of the MIRP connectivity has given many interesting insights and simplifications for the max-SINR case. Further, other performance metrics pertinent to hetnets such as the average fraction of load carried by each tier in the hetnet and the area-averaged rate acheived by an MS that is in coverage in a hetnet can also be derived using the results in this section. We refer the reader to [3, Theorems 2, 3 and 4] for these results.

V. Numerical Examples and Discussion

In this section, we provide some numerical examples that complement the theoretical results presented until now. We restrict ourselves to the study of a two tier hetnet consisting of the macrocell and the femtocell networks, respectively, under the max-SINR connectivity model. Also, please refer to Appendix A for the algorithm to perform the Monte-Carlo simulations. For all the studies in this paper, $\lambda_2 = 5\lambda_1$, $P_1 = 25P_2$, $\varepsilon = 3$, and $\beta_2 = 1$ dB, where the subscripts '1' and '2' correspond to macrocell and femtocell networks, respectively. Further, under the closed-access BS association scheme, the MS has access to the macrocell network only.

In Figures 1, 2 and 3, we study the coverage probability, coverage conditional average rate and the average fraction of users served by the macrocell network, respectively, for various configurations of shadow fading distributions at the macrocell and the femtocell BSs. Note that the expressions for the coverage conditional average rate and the average fraction of users served by the macrocell network can be found in [3, Theorems 2, 3 and 4]. In all the figures, T1 (T2) stand for tier 1, i.e. the macrocell network (tier 2, i.e. the femtocell network). Further, Exp($\cdot$) and LN($\cdot$) are abbreviations for exponential distribution with a given mean and log-normal distribution

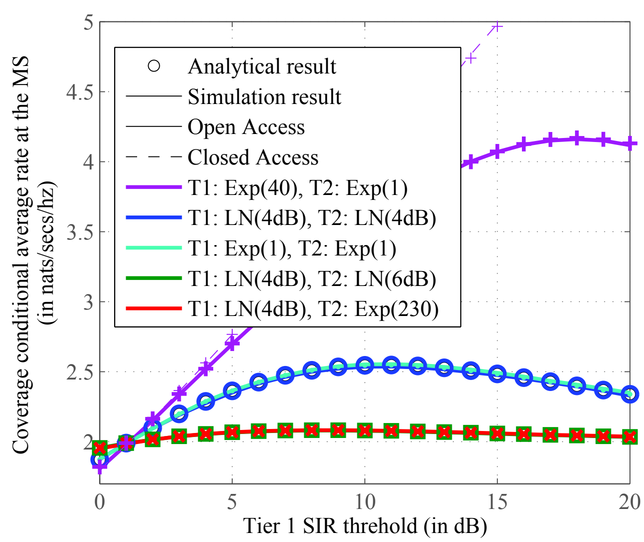

Fig. 3. Two-tier hetnet: Variation of coverage conditional average rate with Tier 1 SIR threshold and different shadow fading distributions

with a zero mean and standard deviation (when the random variable is expressed in dB), respectively, and they represent distribution of the shadow fading factors of the corresponding tiers.

While the expressions in Theorem 3 clearly show that a MS has a better coverage probability under open-access than closed-access, the plots in Figure 1 provides a quantitative justification for the same. The coverage probability curve corresponding to the exponential fading distribution at both the tiers 1 and 2 with means 40 and 1, respectively, also corresponds to the case where $P_1 = 1000P_2$, with the shadow fading factors at both the tiers being unit mean exponential distributions. The open and closed access have approximately the same coverage probabilities because the MS is almost always served by a macrocell BS, as can been seen in the corresponding curve in Figure 2. As a result, blocking access to the femtocell BSs altogether, has only a marginal influence on the coverage probability at the MS.

The two curves following the aforementioned curve in Figures 1-3 complement the fact that all the three performance metrics are identical irrespective of the distribution of the shadow fading factors, when the shadow fading factors have the same distribution across all the tiers. The last two curves in Figures 1-3 show that all the performance metrics are identical as long as the shadow fading coefficients of the corresponding tiers have the same $(2/\varepsilon)^{\text{th}}$ moments. Note that $\mathbb{E}\left[\Psi^{\frac{2}{\varepsilon}}\right]$ is the same when Ψ has a log-normal distribution with zero mean and 6 dB standard deviation or when Ψ is an exponential random variable with mean 230.

A log-normal random variable with zero mean and a given standard deviation is a good model for shadow fading factors. Note that the femtocell network is introduced to improve the indoor performance. The shadow fading factors in the indoor environments are known to have a comparable or greater standard deviation than otherwise. Such a situation is represented by last four curves in Figures 1-3. The gap between the open and closed access coverage probability curves indicate the contribution of the femtocell network in providing coverage to the MS. It is immediately clear that the dense low-power femtocell network has a more critical role in providing coverage in realistic indoor models, when we look at the last four curves in Figures 1 and 2.

Under open-access, the coverage probability and the coverage conditional average rate (see Figures 1 and 2) for all the 5 curves mentioned above intersect when the SIR threshold for the macrocell network is equal to 1 dB. This brings us to an important point that when the SIR threshold is the same for all the tiers, these metrics become independent of the transmission power and shadow fading factors of the different tiers, and collapses to the corresponding metrics in a single-tier network with the same path-loss exponent and SIR threshold. Along the same lines, the coverage conditional average rate for a two-tier hetnet under closed-access also collapses to that of a single-tier network, and is independent of the transmission power and shadow fading factors of the different tiers.

VI. Conclusions

In this paper, for the most general model of the hetnets, the downlink coverage probability and other related performance metrics such as the average downlink rate and average fraction of users served by each tier of the hetnet are characterized for the max-SINR connectivity model. Semi-analytical expression for the hetnet coverage probability is obtained, and several properties pertaining to the hetnet downlink performance are analyzed, which provide great insights about these complex networks. As an example, we identify the MIRP connectivity model to be equivalent to the former model under certain special conditions. These models are much simpler to analyze and the results for these models expose interesting properties of the hetnet. The results in this paper greatly generalize the existing hetnet performance characterization results and are essential for better understanding of the future developments in wireless communications that are heavily based on hetnets.

Appendix

A. Simulation Method

The k^{th} tier of the hetnet with K tiers is identified by the following set of system parameters: $(\lambda_k,\ P_k,\ \Psi_k,\ \varepsilon_k,\ \beta_k)$, where the symbols have all been defined in Section II, and $k = 1,\ 2, \cdots,\ K$, where K is the total number of tiers. Now we illustrate the steps for simulating the hetnet in order to obtain the SINR distribution and the coverage probability assuming the MS is at the origin. The algorithm for the Monte-Carlo simulation is as follows:

1) Generate N_k random variables according to a uniform distribution in the circular region of radius R_B for the locations of all the k^{th} tier BSs, where $N_k \sim \text{Poisson}\left(\lambda_k \pi R_B^2\right)$.

3) Compute the SINR at the desired BS according to Section II-3 and record the tier index I of the desired BS.

Repeat the same procedure T (typically, > 50000) times. Finally, the tail probability of SINR at η is given by $\frac{\{\text{\# of trials where SINR} > \eta\}}{T}$, and the coverage probability is given by $\sum_{k=1}^{K} \frac{\{\text{\# of trials where } I=k \text{ and } SINR>\beta_k\}}{T}$.

References

[1] P. Madhusudhanan, J. G. Restrepo, Y. Liu, T. X. Brown, and K. Baker, "Multi-tier network performance analysis using a shotgun cellular system," in *IEEE Globecom 2011 Wireless Communications Symposium*, Dec. 2011, pp. 1 –6.

[2] P. Madhusudhanan, J. Restrepo, Y. Liu, and T. Brown, "Downlink coverage analysis in a heterogeneous cellular network," in *Global Communications Conference (GLOBECOM), 2012 IEEE*, Dec 2012, pp. 4170–4175.

[3] P. Madhusudhanan, J. G. Restrepo, Y. Liu, and T. X. Brown, "Heterogeneous cellular network performance analysis under open and closed access," in *Global Telecommunications Conference (GLOBECOM 2012), 2012 IEEE*, december 2012, pp. 1 –6.

[4] S.-P. Yeh, S. Talwar, G. Wu, N. Himayat, and K. Johnsson, "Capacity and coverage enhancement in heterogeneous networks," *Wireless Communications, IEEE*, vol. 18, no. 3, pp. 32 –38, june 2011.

[5] A. Damnjanovic, J. Montojo, Y. Wei, T. Ji, T. Luo, M. Vajapeyam, T. Yoo, O. Song, and D. Malladi, "A survey on 3GPP heterogeneous networks," *Wireless Communications, IEEE*, vol. 18, no. 3, pp. 10 –21, june 2011.

[6] A. Damnjanovic, J. Montojo, J. Cho, H. Ji, J. Yang, and P. Zong, "UE's role in lte advanced heterogeneous networks," *Communications Magazine, IEEE*, vol. 50, no. 2, pp. 164 –176, february 2012.

[7] 4G Americas Report. (2011, February) 4G Mobile Broadband Evolution: 3GPP Release 10 and Beyond. [Online]. Available: http://www.4gamericas.org/

[8] Qualcomm. (2010, February) LTE Advanced: Heterogeneous network. [Online]. Available: http://www.qualcomm.com/documents/files/lte-advanced-heterogeneous-networks.pdf

[9] V. Chandrasekhar, J. Andrews, and A. Gatherer, "Femtocell Networks: A Survey," *Communications Magazine, IEEE*, vol. 46, no. 9, pp. 59–67, September 2008.

[10] X. Lagrange, "Multitier cell design," *Communications Magazine, IEEE*, vol. 35, no. 8, pp. 60 –64, aug 1997.

[11] S. Mukherjee, "Downlink SINR distribution in a heterogeneous cellular wireless network with biased cell association," in *IEEE ICC 2012 - 1st International Workshop on Small Cell Wireless Networks*, 2012.

[12] ——, "Distribution of downlink SINR in heterogeneous cellular networks," *IEEE Journal on Selected Areas in Communications*, vol. 30, no. 3, pp. 575 –585, april 2012.

[13] ——, “Downlink SINR distribution in a heterogeneous cellular wireless network with max-SINR connectivity,” in *Communication, Control, and Computing (Allerton), 2011 49th Annual Allerton Conference on*, sept. 2011, pp. 1649 –1656.

[14] H. S. Dhillon, R. K. Ganti, F. Baccelli, and J. G. Andrews, “Modeling and analysis of K-tier downlink heterogeneous cellular networks,” *IEEE Journal on Selected Areas in Communications*, vol. 30, no. 3, pp. 550 –560, april 2012.

[15] ——, “Coverage and ergodic rate in K-tier downlink heterogeneous cellular networks,” in *49th Annual Allerton Conference on Communication, Control, and Computing (Allerton), 2011*, sept. 2011, pp. 1627 –1632.

[16] H. Dhillon, R. Ganti, and J. Andrews, “A tractable framework for coverage and outage in heterogeneous cellular networks,” in *Information Theory and Applications Workshop (ITA), 2011*, feb. 2011, pp. 1 –6.

[17] ——, “Load-aware modeling and analysis of heterogeneous cellular networks,” *Wireless Communications, IEEE Transactions on*, vol. 12, no. 4, pp. 1666–1677, April 2013.

[18] H.-S. Jo, Y. J. Sang, P. Xia, and J. Andrews, “Heterogeneous cellular networks with flexible cell association: A comprehensive downlink sinr analysis,” *Wireless Communications, IEEE Transactions on*, vol. 11, no. 10, pp. 3484–3495, October 2012.

[19] P. Madhusudhanan, J. G. Restrepo, Y. Liu, T. X. Brown, and K. R. Baker, “Downlink performance analysis for a generalized shotgun cellular system,” *CoRR*, 2010. [Online]. Available: http://arxiv.org/abs/1002.3943

[20] V. M. Nguyen and F. Baccelli, “A stochastic geometry model for the best signal quality in a wireless network,” in *Modeling and Optimization in Mobile, Ad Hoc and Wireless Networks (WiOpt), 2010 Proceedings of the 8th International Symposium on*, 31 2010-june 4 2010, pp. 465 –471.

[21] J. F. C. Kingman, *Poisson Processes (Oxford Studies in Probability)*. Oxford University Press, USA, January 1993.

[22] P. Madhusudhanan, J. G. Restrepo, Y. E. Liu, and T. X. Brown, “Downlink analysis for a heterogeneous cellular network,” *to be submitted to Transactions on wireless communications*, 2014. [Online]. Available: http://arxiv.org/abs/1403.7682

[23] P. Madhusudhanan, J. G. Restrepo, Y. Liu, T. X. Brown, and K. Baker, “Stochastic ordering based carrier-to-interference ratio analysis for the shotgun cellular systems,” *Wireless Communications Letters, IEEE*, vol. PP, no. 99, pp. 1 –4, 2012.