Statistical Mechanics Approach for the Detection of Multiple Wireless Sources via a Sensor Network

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***Abstract*—In this paper, we apply statistical mechanics methods to the problem of detection of multiple primary wireless sources by a wireless sensor network. We assume that the location of the primary sources is known, but that the channel connecting them to the sensors is random. The sensor network tries to detect which sources are emitting by employing a belief propagation algorithm. We use the Replica approach to estimate the probability of error and we provide analytical expressions and numerical results for the case of random connectivity between sources and sensor nodes, for the fading channel model. This method can provide a simple way to calculate performance metrics for the detection problem.**

I. Introduction

Dynamic spectrum sensing has emerged as an important issue in the context of heterogeneous, hierarchically structured wireless networks, where secondary users need to sense the presence and transmitted energy of primary (licensed) sources. This ensures that the transmission by secondary, opportunistic sources does not interfere too heavily on the signal of primary sources and vice-versa. The detection and localization of primary sources may be achieved by making use of all available information at the location of secondary sensors. One approach is to measure the delay of each signal received at several sensors and essentially by triangulation to locate the primary sources [1]. However, multiple scattering and the interference due to the reception of multiple measurements from different sources as well as the need of synchronization are important impediments of this method.

The use of non-coherent energy detection [2] is perhaps the simplest approach, since it does not require synchronization and is less sensitive to multiple scattering. In this case, the problem boils down to finding the most probable strength of interference at any location in the network, conditioned on the total interference energy measured at the sensor locations. For this purpose, some information for the statistics of the signal is necessary. One approach is to simply assume that the primary signal is spatially correlated due to the fast and/or slow fading of the channel [3]. Thus, given the signal at the location of the sensors it is possible to obtain the most probable interference level at other locations. This method should provide satisfactory results when the fluctuations due to fading are large. Otherwise, one needs to factor in the effects of pathloss by taking into account that the interference at all locations is due to a discrete number of primary sources located at given points in the network. The outcome of such an algorithm is the most probable locations and power strengths of the primary sources and from that the primary signal at all locations can be obtained. In [4] such an algorithm was developed, providing the powers and locations of primary sources in simple examples with a few sources with reasonable accuracy. However, since the algorithm is centrally controlled, it is not clear how it can work in a large scale system with a fixed source density. Therefore, distributed algorithms are necessary.

A particular family of distributed algorithms that has been known to be near-optimal in some cases is based on message passing [5] and its variants. Besides their efficiency, the appeal of message passing procedures resides in their local nature. Information is propagated along the edges of the graph and each message is updated using other messages coming into the same node, resulting to the possibility of parallelization [6]. In such algorithms the convergence to the most probable result is reached in essentially linear time in the number of sources. Typically the channel matrix is required to be sparse, since this makes the type of exchanged messages simpler. It is typically used in the case of binary unknowns, i.e. when the sources are assumed either on or off. In this case the almost-sure convergence to the most probable result has been shown for both random fully connected [7] as well as sparse (Erdos-Renyi) graphs. They have also been extended to non-binary sources [8], [9]. In particular, this has been shown to be exact for fully connected random graphs [10], [11] under some sparsity constraints.

In this paper we address the performance of message passing in the detection of large-scale network of multiple primary sources from a network of sensors. As a first step we assume that the locations of the primary users are known and that their transmitters are either on or off. This is a realistic scenario, when through radio-neighborhood maps the sensors may know the location of the primary users, but do not know if they are active. Despite the apparent simplicity of the model, it is still quite challenging due to the large scale nature of problem and the fact that each sensor receives a signal from multiple sources. The basic question we would like to address is: For a given type of environment and a REM, what is the region in the parameter space of the system, such as density of sensors and primary sources, signal to noise ratio for the power, where the detection of the active sources is adequate? To answer this question, we applied a statistical physics approach to provide a closed set of equations for the density

This work was funded by the research program 'CROWN', through the Operational Program 'Education and Lifelong Learning 2007-2013' of NSRF, which has been co-financed by EU and Greek national funds.

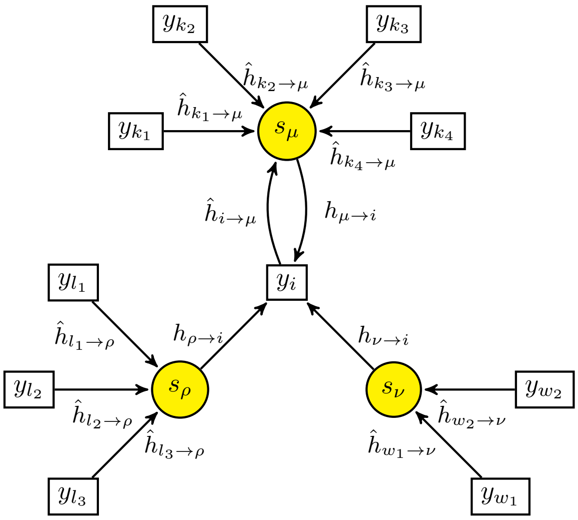


Fig. 1. Graph depicting the incoming messages to source s_μ ($\hat{h}_{\to\mu}$) and sensor y_i ($h_{\to i}$). The network structure has been modelled as a Bethe lattice [13], where every sensor/source is connected to and arbitary number of sources/sensors .

of two parameters, which can be solved straightforwardly using population dynamics. As a result we obtain the average detection probability of the sources. This set of equations corresponds to the so-called replica symmetric solution [12] and is effectively a mean-field result, valid for Erdos-Renyi type graphs. Nevertheless, by comparing this to the results from the behavior of the message passing algorithm in many realizations of a two-dimensional random network, we find good agreement.

The rest of the paper is organized as follows: In section II we present the system model. Section III presents the belief propagation algorithm that is used for detection, while Section IV analyzes the replica symmetric solution for the model. Section V presents the simulation results. Concluding remarks in Section VI end the paper.

II. System Model

We consider a wireless network with sensor nodes and primary source nodes randomly located in the field with average densities N_s and N_t, respectively. The channel power strength at the νth sensor location due to transmitter i is denoted by

$$G_{i\nu} = \frac{P_0}{2}\left(\frac{z_{i\nu} r_0^\gamma}{d_{i\nu}^\gamma + r_0^\gamma}\right) \tag{1}$$

In the above equation, P_0 is the transmission power of source i, assumed known and fixed, $z_{i\nu}$ is the fading coefficient, $d_{i\nu}$ is the distance between the source and the sensor, r_0 is a cutoff distance of the pathloss model and γ is the pathloss exponent. In each instantiation of the network of area A the numbers of sensor and primary nodes are random, following the Poisson distribution with parameters $\lambda_s = N_s A$ and $\lambda_t = N_t A$, respectively. For simplicity however, when generating the network numerically it is convenient to generate a fixed number of sensors and sources equal to $M = \lambda_s A$ and $N = \lambda_t A$ for a unit area.

For pragmatic reasons we assume that a sensor detects a source if $G_{i\nu} \le G_0 = 0.5 P_0 R_c^{-\gamma}$, for a certain radius R_c which depends on the sensitivity of the sensor receiver. The number of connected sensors per source follows the Poisson distribution, whose parameter $\rho_s = N_s A_{\text{eff}}$ is determined in Appendix A by calculating the effective area over which a source can be observed by a sensor which is

$$A_{\text{eff}} = \pi R_0^2 \Gamma\left(1 + \frac{2}{\gamma}\right) \exp\left\{-\frac{r_0^\gamma}{R_c^\gamma}\right\} \tag{2}$$

Similarly, the number of connected sources per sensor has Poisson parameter $\rho_t = N_t A_{\text{eff}}$.

Having discussed the measurement matrix $\boldsymbol{G}$ with entries given by $G_{i\mu}$ we are now in the position to present the detection process. Let $\boldsymbol{\sigma} = [\sigma_1, \sigma_2, ..., \sigma_N]^T$ be the $N \times 1$ column vector, with entries $\sigma_\tau = 2$ or $\sigma_\tau = 0$, depending on whether the corresponding source is transmitting or not. The power measured at each sensor node is then corrupted by noise, which we take to be additive white Gaussian given by the vector $\boldsymbol{\eta}$. The measurement vector at the sensors is $\boldsymbol{w} = \boldsymbol{G}\boldsymbol{\sigma} + \boldsymbol{\eta}$. It is convenient to shift the variables $\sigma_i = s_i + 1$, so the new vector $\boldsymbol{s}$ has elements $s_i = \pm 1$. Then the shifted vector $\boldsymbol{y} = \boldsymbol{w} - \boldsymbol{G}\mathbf{1}$ is given by

$$\boldsymbol{y} = \boldsymbol{G}\boldsymbol{s} + \boldsymbol{\eta} \tag{3}$$

where $\mathbf{1} = [1, 1, \ldots, 1, 1]^T$. It is obvious that the measurement matrix $\boldsymbol{G}$ has a sparse form. We will use this fact to make a key approximation later on that there are no (short) loops in the corresponding graph, thus approximating the graph by a tree.

In order to recover the signal $\boldsymbol{s}$, we will solve the LS minimization problem which can be formulated as follows

$$\hat{\boldsymbol{s}} = \arg \min_{\boldsymbol{s} \in [-1,1]^n} \|\boldsymbol{y} - \boldsymbol{G}\boldsymbol{s}\|^2 \tag{4}$$

Before defining the message passing algorithm in the next section, we will define the above quantity as an energy function

$$\begin{aligned} \mathcal{E} &= \sum_i \varepsilon_i(s_{\partial i}) \\ &= \sum_i \left(y_i - \sum_{\nu \in \partial i} G_{i\nu} s_\nu\right)^2 \end{aligned} \tag{5}$$

where ε_i is the energy (cost) corresponding to the i-th sensor node and denoting as ∂i ($\partial \mu$) the set of sources (respectively sensor nodes) adjacent to sensor node i (respectively source node μ).

The above detection problem can be depicted in Figure 1. To proceed we recast the detection problem as a statistical mechanics system with energy $\mathcal{E}$ and inverse temperature β. We define the so-called partition function $\mathbb{Z}$ as

$$\mathbb{Z} = \sum_{s_\mu} \exp\{-\beta \mathcal{E}\} \tag{6}$$

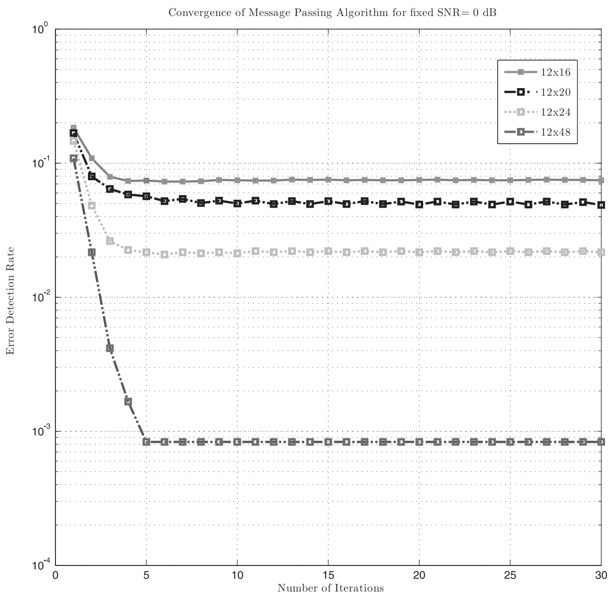


Fig. 2. Convergence of the Message Passing algorithm in random networks with various connectivity matrices for fixed SNR $= 0$ dB.

and the corresponding Helmholtz free energy as $\mathbb{F} = -\beta^{-1}\ln\mathbb{Z}$. Then the detection problem above becomes equivalent to finding the minimum energy configuration of the statistical mechanics system, which can be obtained by taking the $\beta\to\infty$ limit. As noted above, we will only consider the special case of binary variables, but the same arguments with some small modifications can be used to extend it to the field of continuous variables.

III. The Belief Propagation Algorithm

We will now describe the belief propagation method that will minimize our cost function in (5).

If we denote by $m_{\mu\to i}(s_\mu)$ and $\hat{m}_{i\to\mu}(s_\mu)$ the incoming messages to the sensor node y_i and source s_μ respectively, we get the following update rules

$$m_{\mu\to i}(s_\mu) \quad\propto\quad \prod_{j\in\partial\mu\backslash i}\hat{m}_{j\to\mu}(s_\mu) \tag{7}$$

and

$$\begin{aligned}\hat{m}_{i\to\mu}(s_\mu) \quad&\propto\quad \sum_{s_\mu} f_i(s_{\partial i})\prod_{\nu\in\partial i\backslash\mu} m_{\nu\to i}(s_\nu) \\ f_i(s_{\partial i}) \quad&=\quad \exp\{-\beta\varepsilon_i\}\end{aligned} \tag{8}$$

Since all messages are functions of binary variables they can be parametrized by their log-likelihood ratios (usually called “effective (local) field”) as,

$$\hat{h}_{i\to\mu} \equiv \frac{1}{2\beta}\cdot\log\left[\frac{\hat{m}_{i\to\mu}(s_\mu=+1)}{\hat{m}_{i\to\mu}(s_\mu=-1)}\right] \tag{9}$$

so that

$$\hat{m}_{i\to\mu}(s_\mu) = \frac{1}{2}\left(1+s_\mu\tanh(\beta\hat{h}_{i\to\mu})\right) \tag{10}$$

and correspondingly

$$m_{\mu\to i}(s_\mu) = \frac{1}{2}\left(1+s_\mu\tanh(\beta h_{\mu\to i})\right) \tag{11}$$

To include the bias due to the prior distribution of s_μ^0 for each source μ it is convenient to first perform the following gauge transformation

$$s_\nu \mapsto \tilde{s}_\nu s_\nu^0 \tag{12}$$

Now, the optimum values of $\tilde{s}$ are simply $\tilde{s}=1$. In order to maintain the energy function of (5) invariant to the aforementioned transformation, we also need to transform the gain matrix $\boldsymbol{G}$ as follows

$$\boldsymbol{G}\mapsto\boldsymbol{G}\boldsymbol{S}^0 \tag{13}$$

where $\boldsymbol{S}^0 = \text{diag}\left(s_\nu^0\right)$.

Rearranging the various summations and using the transform described above in (12) and (13) we get the expression of the update rules in a recursive way (with each node's prior distribution included) as follows:

$$m_{\mu\to i}(\tilde{s}_\mu) = \frac{1}{Z^{\mu\to i}}\prod_{j\in\partial\mu\backslash i}\hat{m}_{j\to\mu}(\tilde{s}_\mu) \tag{14}$$

This equation may be restated in terms of the fields $\hat{h}$ and h as

$$h_{\mu\to i} = \sum_{j\in\partial\mu\backslash i}\hat{h}_{j\to\mu} \tag{15}$$

The corresponding equation for $\hat{m}_{i\to\mu}$ is

$$\hat{m}_{i\to\mu}(\tilde{s}_\mu) = \frac{1}{Z^{i\to\mu}}\sum_{\tilde{s}_{\nu\partial i\backslash\mu}} f_i(\tilde{s}_{\partial i})\prod_{\nu\in\partial i\backslash\mu} m_{\nu\to i}(\tilde{s}_\nu) \tag{16}$$

where

$$f_i(\tilde{s}_{\partial i}) = \exp\left\{-\beta\left(y_i-\sum_{\nu\in\partial i}G_{i\nu}\tilde{s}_\nu s_\nu^0\right)^2\right\} \tag{17}$$

After some algebra it is easy to show that

$$\hat{m}_{i\to\mu}(s_\mu) = \frac{1}{2}\left(1+\tilde{s}_\mu\mathsf{t}_\beta\right) \tag{18}$$

where

$$\mathsf{t}_\beta = \frac{\mathsf{s}_\beta}{\mathsf{c}_\beta} \tag{19}$$

with

$$\begin{aligned}\mathsf{c}_\beta \quad&=\quad \sum_{\tilde{s}_{\nu\in\partial i}}\exp\left\{-\beta\left[y_i-\sum_{\nu\in\partial i}G_{i\nu}\tilde{s}_\nu s_\nu^0\right]^2\right\} \\ &\cdot\quad \prod\nolimits_{\nu\in\partial i\backslash\mu}\frac{\exp\{\beta\tilde{s}_\nu h_{\nu\to i}\}}{2\cosh(\beta h_{\nu\to i})}\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathsf{s}_\beta \quad&=\quad \sum_{\tilde{s}_{\nu\in\partial i}}\hat{s}_\mu\exp\left\{-\beta\left[y_i-\sum_{\nu\in\partial i}G_{i\nu}\tilde{s}_\nu s_\nu^0\right]^2\right\} \\ &\cdot\quad \prod\nolimits_{\nu\in\partial i\backslash\mu}\frac{\exp\{\beta\tilde{s}_\nu h_{\nu\to i}\}}{2\cosh(\beta h_{\nu\to i})}\end{aligned} \tag{21}$$

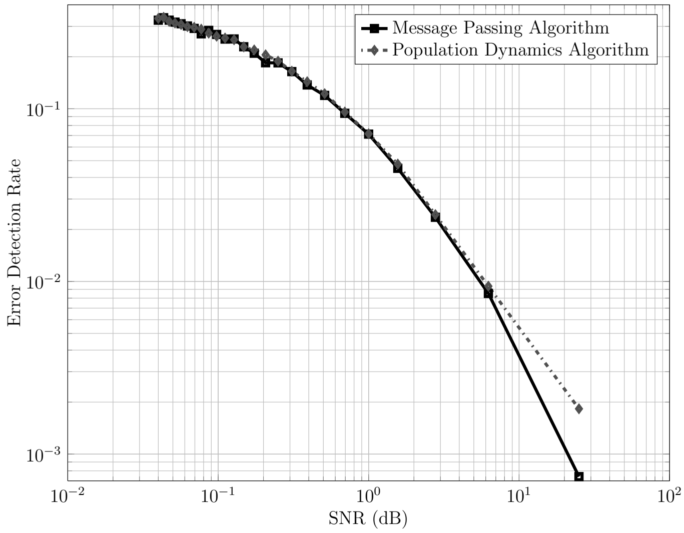


Fig. 3. Graphical comparison of Message Passing algorithm and Population Dynamics method in an irregular bipartite graph. The network we deployed for testing these two algorithms, consisted of 10 sources and 10 sensors with a random degree of connectivity between them. It is readily seen that both algorithms give almost the same results, regarding error detection rate. In this case, we considered a tentative environment without pathloss but with Rayleigh fading. An important factor that we neglected, on purpose, is the loop-free tree and it is definitely an explanation for the small difference of these two curves in high SNR values.

These equations are quite complicated to update in the message passing algorithm. However, as mentioned in the previous section, we are ultimately interested in their $\beta \to \infty$ limit. In this case, the corresponding equation for $\hat{h}_{i\to\mu}$ is derived in the Appendix to be

$$\hat{h}_{i\to\mu} = \sigma_\mu \frac{\xi^+ - \xi^-}{2} \tag{22}$$

where

$$\xi^+ = \min_{s_{\nu\in\partial i}} \left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\backslash\mu} h_{i\to\nu} \right) \tag{23}$$

$$\sigma^\mu = \arg\min_{s_{\nu\in\partial i}} \left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\backslash\mu} h_{i\to\nu} \right) \tag{24}$$

$$\xi^- = \min_{s_{\nu\in\partial i\backslash\mu}, s_\mu = -\sigma_m u} \left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\backslash\mu} h_{i\to\nu} \right) \tag{25}$$

As we see, $-\beta\xi^+$ is the maximum of all exponents in (20), and σ^μ is the value of s_μ in the corresponding $\boldsymbol{s}$ vector, while $-\beta\xi^-$ is the maximum over the subset of the exponents with $s_\mu = -\sigma^\mu$. The above equations (15) and (22) form the basis of our message passing algorithm.

IV. The Bethe Approximation

It is well known that the above belief propagation equations are the stationary points of the so-called Bethe free energy, which effectively only takes into account pairwise interactions between nodes [14]. This approximation becomes exact when there are no loops in the graph of the network. In this approximation, we can write the free energy of the system as the sum of the contributions of sensor and source sites as well as bond contributions,

$$\mathbb{F}_{Bethe}[\{h_{\mu\to i}\}, \{\hat{h}_{i\to\mu}\}] = \sum_\mu \mathbb{F}_\mu + \sum_i \mathbb{F}_i - \sum_{(i,\mu)} \mathbb{F}_{i\mu} \tag{26}$$

where the contribution of the source sites μ and sensor sites i are up to a constant respectively given by

$$\mathbb{F}_\mu = \ln\left[\sum_{s_\mu}\prod_{j\in\partial\mu}\left(1 + s_\mu \tanh(\beta\hat{h}_{j\to\mu})\right)\right] \tag{27}$$

and

$$\mathbb{F}_i = \ln\left[\sum_{s_\nu} e^{-\beta\varepsilon_i} \prod_{\nu\in\partial i\backslash\mu} \left(1 + s_\mu \tanh\left(\beta h_{\nu\to i}\right)\right)\right] \tag{28}$$

and the contribution from the bond $i \leftrightarrow \mu$ is

$$\mathbb{F}_{i\mu} = \ln\left[\frac{1}{2}\left(1 + \tanh(\beta\hat{h}_{i\to\mu})\tanh(\beta h_{\mu\to i})\right)\right] \tag{29}$$

We can use this formulation to obtain an expression for the detection error rate of the system in terms of the fields h and $\hat{h}$. We start by adding an effective magnetic field term in the cost function, $\mathcal{E}$, from (5)

$$\mathcal{E} \to \mathcal{E} - \delta\sum_\mu s_\mu s_\mu^0 = \mathcal{E} - \delta\sum_\mu \tilde{s}_\mu \tag{30}$$

By taking the derivative of $\mathbb{F}$ with respect to δ at $\delta = 0$ in (26), we see that the fraction of of which the state has been detected erroneously is

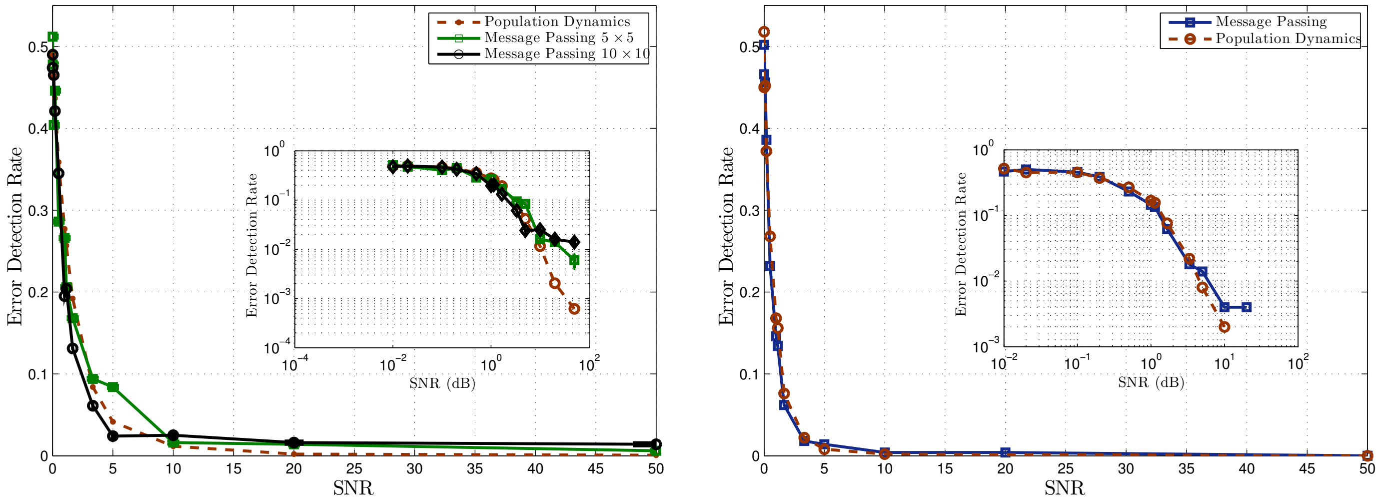
$$\begin{aligned}\mathbb{P}_{\text{err}} &= \frac{1}{2}\left(1 - \frac{1}{M}\sum_\mu \left\langle s_\mu s_\mu^0\right\rangle\right) \\ &= \frac{1}{2}\left(1 - \frac{1}{M\beta}\frac{\partial\mathbb{F}_{Bethe}}{\partial\delta}\bigg|_{\delta=0}\right)\end{aligned} \tag{31}$$

Thus, including the above term in the free energy above alters $\mathbb{F}_\mu$ to

$$\begin{aligned}\mathbb{F}_\mu = \ln\Big[& e^{-\beta\delta}\prod_{j\in\partial\mu}\left(1 - \tanh(\beta\hat{h}_{j\to\mu})\right) \\ & + e^{\beta\delta}\prod_{j\in\partial\mu}\left(1 + \tanh(\beta\hat{h}_{j\to\mu})\right)\Big]\end{aligned} \tag{32}$$

which then gives

$$\begin{aligned}1 - \mathbb{P}_{\text{err}} &= \frac{1}{2M}\sum_\mu \tanh\left(\beta\sum_{i\in\partial\mu}\hat{h}_{i\to\mu}\right) \\ &= \frac{1}{2M}\sum_\mu \tanh\left\{\beta\left(\hat{h}_{i\to\mu} + h_{\mu\to i}\right)\right\} \\ &\underset{\beta\to\infty}{\to} \frac{1}{2M}\sum_\mu \text{sign}\left(\hat{h}_{i\to\mu} + h_{\mu\to i}\right)\end{aligned} \tag{33}$$



(a)

(b)

Fig. 4. Average error detection rate vs. SNR, using the belief propagation algorithm in 100 random networks compared to the population dynamics algorithm in linear and logarithmic axes. The graph corresponds to (a) random networks with 5 sensors and 5 sources, 10 sensors and 10 sources and (b) 10 sensors and 5 sources, with random connectivity. In Figure (a) for comparison purpose only, we simulated the network with 10 sensors and 10 sources with each sensor having effective radius $R_c = 0.5$ and the network with 5 sensors and 5 sources with each sensor having effective radius $R_c = \sqrt{0.5}$ so as to have comparable performance. Additionally r_0 was set to 0.9 for every deployed network. The channel matrix contains in both cases, the pathloss exponent and the Rayleigh fading with $\gamma = 4$.

A. The Replica-Symmetric Approach

The above equations are still dependent on the particular graph realization of the network. In this section we will make one further approximation, namely that the graph is statistically equivalent. Hence the environment of each node or sensor is equivalent with every other. We may therefore define the random fields H and $\hat{H}$ corresponding to the fields $h_{i\to\mu}$ and $h_{\mu\to i}$ above and provide them with distributions $Q(H)$ and $\hat{Q}(\hat{H})$ respectively. These distributions will then be obtained self-consistently by enforcing the message passing equations.

From Eqs. (15) and (22) we get two basic recursive relations for the probability density [15]

$$\hat{Q}\left(\hat{H}\right) = \left\langle \prod_{i=1}^{r} \int dH_i Q(H_i) \cdot \boldsymbol{\delta}\left(\hat{H} - \beta^{-1}\tanh^{-1}\left[\mathrm{t}_\beta(H)\right]\right) \right\rangle \tag{34}$$

with the expectation over $\boldsymbol{G}, \boldsymbol{y}, r$ and

$$Q\left(H\right) = \left\langle \prod_{k=1}^{t-1} \int d\hat{H}_k \hat{Q}\left(\hat{H}_k\right) \cdot \boldsymbol{\delta}\left(H - \sum_{k=1}^{t-1} \hat{H}_k\right) \right\rangle_t \tag{35}$$

which are the saddle point equations resulted from the Bethe free energy. Expectations over r and t which are the number of sources per sensor and vice versa are taken over the corresponding distribution coming out from fixed density of sources and sensors in a network.

Generally the above equations cannot be solved in closed form. However, it is possible to obtain the fixed point distributions $Q(H)$ and $\hat{Q}(\hat{H})$ using a simple stochastic algorithm called population dynamics [12], [16]. Instead of working with $Q\left(H\right)$ and $\hat{Q}(\hat{H})$ directly, we represent the effective field distributions by a large population of K copies (fields) randomly

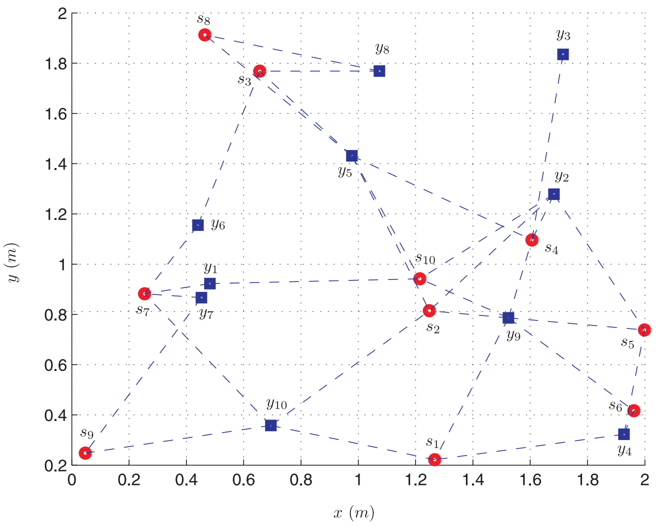


Fig. 5. A random network of 10 sources and 10 sensors in a 2×2 area. The connectivity between the nodes is formed in a completely random way.

drawn from these two distributions. K is sufficiently large (e.g. $K = 1000$) so as to provide good resolution in the desired performance measures [16] We initiate the process by creating K random values for H and K random samples for $\hat{H}$, for example with $K = 1000$. We then continue by generating the number t from a given distribution, in our case the Poisson distribution with parameter ρ_s. We then take $t-1$ random samples from the $\hat{H}$ bin and save their sum at a random location in the H bin. We then generate the number r randomly from a distribution, in our case the Poisson distribution with parameter ρ_t and pull r random samples from the H bin. With these, we calculate the function $\tanh^{-1}[\mathrm{t}_\beta(H)]$ and pass its value to a randomly chosen element in the $\hat{H}$ bin. We continue this process until the distributions of $\hat{H}$ and H converge.

The probability of erroneous detection can be obtained directly from the above distributions (35) and (34) and it is

$$\mathbb{P}_{\text{err}} = \frac{1}{2}\left(1 - \left\langle \operatorname{sign}(H + \hat{H}) \right\rangle_{H,\hat{H}}\right) \tag{36}$$

It should be pointed out that the above error probability includes only the situations where each source is connected to at least one sensor. Therefore to describe the source outage concisely we need to include the probability where a primary source is completely detached from the sensor network $\mathbb{P}_{\text{unc}} = \exp -\rho_s$. Then the total error detection rate is given by

$$\mathbb{P}_{\text{total}} = \mathbb{P}_{\text{unc}} + (1 - \mathbb{P}_{\text{unc}}) \cdot \mathbb{P}_{\text{err}} \tag{37}$$

At this point it is worth to summarize the main aim and basic notion of the proposed algorithm. First, the goal of this paper is to develop an analytic methodology, which can quantify the ability of the message passing algorithm to correctly detect primary sources in a large random wireless sensor network, subject only to the statistical characteristics of the network. This methodology can be described as follows. Given a density of primary users and secondary users N_t and N_s, respectively, the characteristics of propagation, such as the path-loss exponent γ and the fading distribution the coefficients ρ_t and ρ_s (see (2)) can be determined. Note that in this paper we only dealt with Rayleigh fading, but this can be generalized. These coefficients determine the Poisson distributions of the number of connected sources per sensor and the number of sensors per source respectively. These distributions characterize accurately the degree distribution in the actual two-dimensional network. They are then used to characterize the degree distributions of an equivalent irregular random bipartite network. Equations (35) and (34) then determine the distributions $Q(H)$ and $\hat{Q}(\hat{H})$ of the effective fields for such a graph with thermal noise in the detection included, from which finally the probability of detection error is evaluated in (36) and then (37). In the above context the main approximation we have made is that the two dimensional network of sources and sensors can be approximated with a random graph with connections possible between any pair of sensor-source.

V. Simulation Results

In this section we will briefly discuss the numerical validation of the above approach. As a starter we show that message passing is a fast converging algorithm in this context. Indeed, Fig. 2 shows the simulation results regarding the convergence rate for the case of the Message Passing algorithm. It can be seen that for various neighbouring matrices the proposed algorithm converges very fast to the optimal solution. The neighbouring matrices differ not only in the size, which reveals the number of sensors and sources but also in the connectivity between sensors and sources. Without loss of generality, we kept a fixed SNR value of 0 dB. In terms of the complexity, the belief propagation algorithm has the significant advantage of linear complexity, which makes it especially appealing in practice.

Next, we wish to validate the accuracy of the replica symmetric approach depicted in (35) and (34) and the corresponding population dynamics algorithm in comparison to an Erdos-Renyi random graph. In Figure 3 we show the detection error rate for the two employed algorithms, the Message Passing and the Population Dynamics is depicted. It is readily seen that in this case these algorithms have extremely good agreement of performance.

To quantify now the behavior of the replica symmetric approach in two dimensional networks we have generated a number of random instantiations of a network with 5×5, 10×10 and 10×5 sensors and sources respectively. One particular realization is shown in Fig. 5. For each of these networks we calculated the average detection error rate as a function of the noise (or SNR) and then compared it to the corresponding values of the replica-symmetric system obtained using population dynamics. This comparison is depicted in Fig. 4, which shows good agreement between the two algorithms.

Finally, having justified the usefulness of the replica-symmetric methodology, we obtain the values of the detection error rate using population dynamics for various values of ρ_t and ρ_s in Fig. 6. In the left-and side we plot the probability of error given that the source is connected to some sensor, i.e. $\mathbb{P}_{\text{err}}$. We find here the approximate symmetry around the line $\rho_t = \rho_s$; above that the error is higher, while below that the error is much lower. The right figure shows the total value of error $\mathbb{P}_{\text{tot}}$. We see that for this set of parameter values the detection rate is dominated by the probability of non-connection.

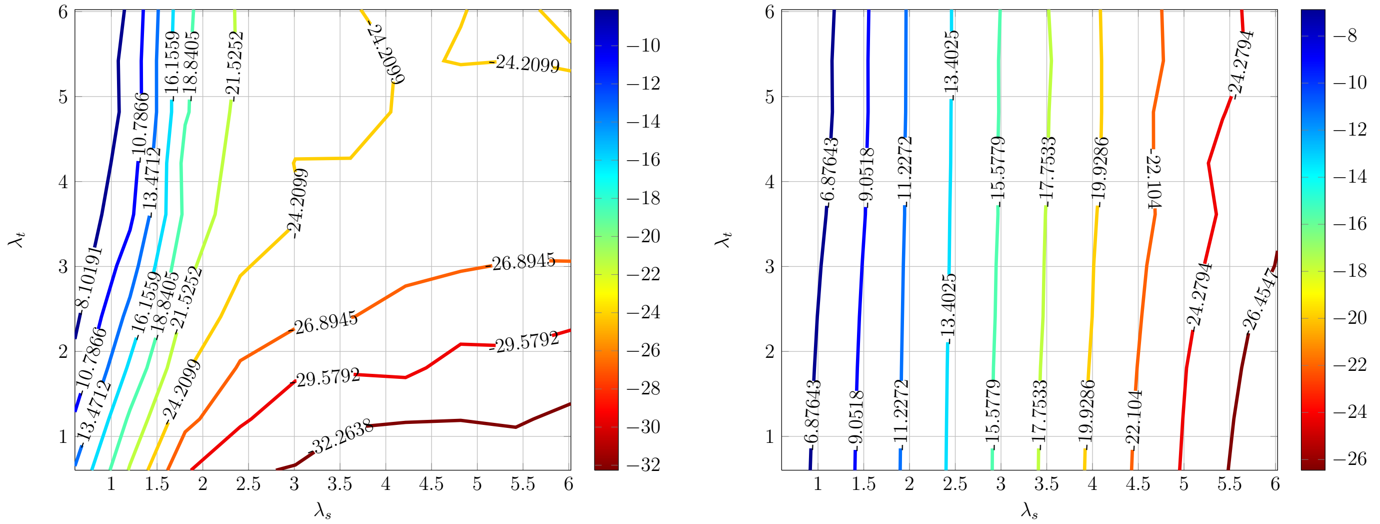
In closing, it is worth saying that although we modelled our network as a Cayley tree, in many of our random networks' simulations, the neighbouring matrices contained many short loops which caused local correlations within groups of elements. This phenomenon did not degrade substantially the overall network performance.

VI. Discussion and Conclusions

This paper presented a study of multiple source detection using a sensor network in the presence of interference and noise. We used the replica symmetric approach to calculate the detection error for the network and showed that its results are in numerical agreement with the more network-message passing approach. This enables us to provide predictions for the detection error behavior in large scale networks, based on only a few parameters of the statistical characteristics of the connections, such as sensor and source density, pathloss exponent and fading. In the future we will expand this algorithm in the prediction of the powers of the sources.

Appendix A
Connectivity Statistics

In this paper we are dealing with a random sensor deployment with the received power in a sensor due to a given



(a) (b)

Fig. 6. Error detection rate as a function of source's ρ_t and sensor's ρ_s factors for fixed SNR $= 10$. Figure (a) depicts the $\mathbb{P}_{\text{err}}$ derived from the population dynamics algorithm according to Eq. (36) and (b) the total error detection rate $\mathbb{P}_{\text{tot}}$ where the probability of a node to be isolated has been considered. We have assumed that $R_c = 2$, $r_0 = 1$ and the environmental factor $\gamma = 4$. The regions with the same error detection rate (in dB) are marked with the same colour. As expected, since ρ_s and ρ_t are proportional to densities N_s and N_t, the total error detection rate is higher when the number of sensors placed within the area of an emitting source is small.

source given by (1). For convenience we use a function, $f(r)$, to represent the edge appearing probability as a function of the distance between two nodes [17]. This link probability function, is very useful since the probability of a sensor to be connected with a source is clearly range dependent. For the Rayleigh fading environment, the channel model has a random component which is exponentially distributed and can be viewed as the pdf of the received power in the sensor. Then, the communication range r can be defined as the distance at which the SNR falls below a certain threshold $P_{th} = R_c^{-\gamma}$,

$$f(r) = \mathbb{P}\left(P_{\text{tx}} \geq P_{\text{th}}|r\right) = \exp\left\{-\frac{r^\gamma + 1}{R_c^\gamma}\right\} \tag{38}$$

For the Rayleigh fading environment the effective area A_{eff} over which the sensor is connected to the sensor is equal to

$$\int_0^\infty 2\pi r f(r) dr = \pi R_c^2 \Gamma\left(\frac{2}{\gamma} + 1\right) \tag{39}$$

Since the position of nodes around a given node is random and independent it can be shown easily that the number of sensors in each finite sub area A_{eff} follows a Poisson distribution [18]

$$\mathbb{P}(k \text{ sensors in } A_{eff}) = \frac{(A_{eff} N_s)^k}{k!} \exp\{-A_{eff} N_s\}$$

and derive the probability that the sensor remains unconnected with every source, as $\mathbb{P}_{\text{unc}} = \exp\{-A_{eff} N_s\}$.

Appendix B
Derivation of (22)

We can write the two terms s_β and c_β by analyzing them in a sum of exponential factors, as,

$$\mathsf{s}_\beta = \sum_k s_\mu \exp\{-\beta \xi_k\} \tag{40}$$

and

$$\mathsf{c}_\beta = \sum_k \exp\{-\beta \xi_k\}. \tag{41}$$

where s_μ is the sign of each the spin we are interested in, the exponents $-\beta\xi_k$ are the exponents in the expression of the sum in (20). The index k goes over all 2^r realizations of the $r = |\partial i|$ sources connected with sensor i. In order to calculate the quantity $\beta^{-1}\tanh^{-1}[\mathsf{t}_\beta(H)]$ in the $\beta \to \infty$ limit we denote

$$\xi^+ = \min_{s_{\nu\in\partial i}} \left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\backslash\mu} h_{i\to\nu}\right) \tag{42}$$

$$\sigma^\mu = \arg\min_{s_{\nu\in\partial i}} \left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\backslash\mu} h_{i\to\nu}\right) \tag{43}$$

In the above σ_μ is the value of s_μ in the configuration with the minimum value of ξ_k, which is ξ^+. Without loss of generality let us denote this realization as $k = 0$.

Thus, we can write the exponential terms as

$$\begin{aligned}\mathsf{s}_\beta &= \sigma^\mu \exp\{-\beta\xi^+\} + \sum_{k\neq 0} s_\mu \exp\{-\beta\xi_k\} \\ &= \exp\{-\beta\xi^+\}\left(\sigma^\mu + \underbrace{\sum_{k\neq 0} s_\mu \exp\{-\beta\xi_k + \beta\xi^+\}}_{\mathcal{A}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathsf{c}_\beta &= \exp\{-\beta\xi^+\} + \sum_{k\neq 0} \exp\{-\beta\xi_k\} \\ &= \exp\{-\beta\xi^+\}\left(1 + \underbrace{\sum_{k\neq 0} \exp\{-\beta\xi_k + \beta\xi^+\}}_{\mathcal{B}}\right)\end{aligned}$$

so that $\tanh^{-1}[\mathsf{t}_\beta(H)] = \tanh^{-1}\left(\frac{\sigma_0+\mathcal{A}}{1+\mathcal{B}}\right)$ and after some algebra we finally get

$$\tanh^{-1}\left(\frac{\sigma^\mu+\mathcal{A}}{1+\mathcal{B}}\right) = \frac{1}{2}\ln\left(\frac{1+\mathcal{B}+\sigma^\mu+\mathcal{A}}{1+\mathcal{B}-\sigma^\mu-\mathcal{A}}\right) \tag{44}$$

However, we would like to know the exact next term that prevails along with ξ^+. So, in case where $\sigma^\mu = +1$ we see that the leading term will come from $(\mathcal{B}-\mathcal{A})$ in the denominator. We define ξ^- as

$$\xi^- = \min_{s_{\nu\in\partial i\setminus\mu}, s_\mu=-1}\left(\varepsilon_i(s_{\nu\in\partial i}) - \sum_{\nu\in\partial i\setminus\mu} h_{i\to\nu}\right) \tag{45}$$

For simplicity, let us define

$$\chi_\mu = \begin{cases} 0, & \text{if} \quad s_\mu = -1 \\ 1, & \text{if} \quad s_\mu = +1 \end{cases} \tag{46}$$

and we finally get

$$\begin{aligned}\tanh^{-1}[\mathsf{t}_\beta(H)] &= \frac{\beta}{2}\left(\xi^+ - \xi^-\right) \\ &+ \ln\left(\frac{1+\sum_{k\neq 0}\chi_\mu\exp\{\beta(\xi^+-\xi_k)\}}{1+\sum_{k\neq 0,\rho}\chi_\mu\exp\{\beta(\xi^--\xi_k)\}}\right)\end{aligned}$$

where ρ is the index of the respective ξ^- term that prevails along with ξ^+. Similarly, for the case of $\sigma^\mu = -1$ and

$$\begin{aligned}\tanh^{-1}[\mathsf{t}_\beta(H)] &= \frac{\beta}{2}\left(\xi^- - \xi^+\right) \\ &+ \ln\left(\frac{1+\sum_{k\neq 0,\rho}\chi_\mu\exp\{\beta(\xi^--\xi_k)\}}{1+\sum_{k\neq 0}\chi_\mu\exp\{\beta(\xi^+-\xi_k)\}}\right)\end{aligned}$$

Therefore, in both cases we finally obtain

$$\beta^{-1}\tanh^{-1}[\mathsf{t}_\beta(H)] \approx \frac{\sigma^\mu}{2}\left(\xi^+ - \xi^-\right) \tag{47}$$

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