Decidable Properties for Regular Cellular Automata

Pietro Di Lena

Department of Computer Science, University of Bologna, Mura Anteo Zamboni 7, 40127 Bologna, Italy. dilena@cs.unibo.it

Abstract. We investigate decidable properties for regular cellular automata. In particular, we show that regularity itself is an undecidable property and that nilpotency, equicontinuity and positively expansiveness became decidable if we restrict to regular cellular automata.

1 Introduction

Cellular Automata (CA) are often used as a simple model for complex systems. They were introduced by Von Neumann in the forties as a model of self-reproductive biological systems [16]. Mathematical theory of CA was developed later by Hedlund in the context of symbolic dynamics [7].

To a cellular automaton one associates the shift spaces generated by the evolution of the automaton on suitable partitions of the configuration space. Adopting Kůrka's terminolgy we call *column subshifts* this kind of shift spaces (see [12] chapter 5). A general approach to the study of a cellular automaton is to study the complexity of its column subshifts (see [5, 13, 10]).

Regularity has been introduced by Kůrka for general dynamical systems [14]. A CA is regular if every column subshift is sofic, i.e. if the language of every column subshift is regular. Kůrka classified CA according to the complexity of column subshift languages [13]. In Kůrka's classification the main distiction is whether the cellular automaton is regular or not. He compared language classification with two other famous CA classifications such as equicontinuity and attractor classification.

In this paper we study the decidability of topological properties for CA. In particular, we show that regularity is not a decidable property (Theorem 7) which implies that the membership in Kůrka's language classes is undecidable. In contrast, we show that some topological properties which are in general undecidable become decidable if we restrict to the class of regular CA. For instance, we show that for regular CA nilpotency, equicontinuity and positively expansiveness are decidable properties (Theorem 6). Moreover, we provide an answer to a question raised in [3] showing that the topological entropy is computable for one-sided regular CA (Theorem 5).

The paper is organized as follows. Section 2 is devoted to the introduction of the notation and general definitions while Section 3 contains our results.

2 Notations and Definitions

2.1 Shift Spaces and representations of Sofic Shifts

Let $A = \{a_1, ..., a_n\}$ be a finite alphabet, $n > 1$. For any $k > 0$, $w_1 w_2 ... w_k \in A^k$ is a finite sequence of elements of A. The sets $A^{\mathbb{Z}}$ and $A^{\mathbb{N}}$ are respectively the set of doubly infinite sequences $(x_i)_{i \in \mathbb{Z}}$ and mono infinite sequences $(x_i)_{i \in \mathbb{N}}$ where $x_i \in A$. Let $x \in A^{\mathbb{Z}}$, for any integer interval $[i, j]$, $x_{[i,j]} \in A^{j-i+1}$ is the finite subword $x_i x_{i+1} ... x_j$ of x.

Define the metric d on $A^{\mathbb{Z}}$ by $d(x, y) = \Sigma_{i \in \mathbb{Z}} \frac{d_i(x_i, y_i)}{2^{|i|}}$ where $d_i(x_i, y_i) = 1$ if $x_i = y_i$ and $d_i(x_i, y_i) = 0$ otherwise. The set $A^{\mathbb{Z}}$ endowed with metric d is a compact metric space. A *dynamical system* is a pair (X, F) where $F : X \to X$ is a continuous function and X is a compact metrizable space. The *shift* map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by $\sigma(x)_i = x_{i+1}$, is an homeomorphism of the compact metric space $A^{\mathbb{Z}}$. The dynamical system $(A^{\mathbb{Z}}, \sigma)$ is called *full n-shift* or simply *full shift*.

A *shift space* or *subshift* (X, σ) is a closed shift invariant subset of $A^{\mathbb{Z}}$ endowed with σ. The shift dynamical system (X, σ) is called one-sided if $X \subseteq A^{\mathbb{N}}$. In general, we denote the subshift (X, σ) simply with X.

Let denote with $\mathcal{B}_k(X) = \{x \in A^k \mid \exists y \in X, \exists i \in \mathbb{Z}, y_{[i,i+k-1]} = x\}$ the set of *allowed k-blocks* of the subshift X, $k > 0$. The language associated to a subshift X is denoted with $\mathcal{L}(X) = \Sigma_{i \in \mathbb{N}} \mathcal{B}_k(X)$. Any subshift is completely determined by its language (see [15]). The language of a subshift X is:

1. *factorial*: if $xyz \in \mathcal{L}(X)$ then $y \in \mathcal{L}(X)$.
2. *extendable*: $\forall x \in \mathcal{L}(X)$, $\exists y \in \mathcal{L}(X)$ such that $xy \in \mathcal{L}(X)$.

The language $\mathcal{L}(X)$ of a subshift X is *bounded periodic* if there exists integers $m \geq 0$, $n > 0$ such that $\forall x \in \mathcal{L}(X)$ and $\forall i \geq m$, $x_i = x_{i+n}$.

A *factor map* $F : (X, \sigma) \to (Y, \sigma)$ is a continuous and σ-commuting function, i.e. $F \circ \sigma = \sigma \circ F$. If F is *onto* (or *surjective*), X is called *extension* of Y and Y is called *factor* of X. If F is biiective, it is a *topological conjugacy* and X, Y are said to be *topologically conjugated* shift spaces.

A subshift is *sofic* if it can be represented by means of a *labeled graph*. We review the representation of a sofic shift as *vertex shift* of a labeled graph. A labeled graph $\mathcal{G} = (V, E, \zeta)$ consists of a set of *vertices* V, a set of *edges* E and a *labeling function* $\zeta : V \to A$ which assigns to each vertex $v \in V$ a symbol from a finite alphabet A. Each edge $e \in E$ identifies an initial vertex $i(e) \in V$ and a terminal vertex $t(e) \in V$. We denote the existence of an edge between vertices $v, v' \in V$ by $v \to v'$. Every sofic shift can be represented as the set of (mono or doubly) infinite sequences generated by the labels of vertices of a labeled graph. That is, the labeled graph $\mathcal{G} = (V, E, \zeta)$, with $\zeta : V \to A$, represents the (two-sided) sofic shift

$$S_{\mathcal{G}} = \{x \in A^{\mathbb{Z}} \mid \exists (v_i)_{i \in \mathbb{Z}} \in V^{\mathbb{Z}}, v_i \to v_{i+1}, \zeta(v_i) = x_i, i \in \mathbb{Z}\}.$$

The topological entropy $h(X) = \lim_{n\to\infty} \log|\mathcal{B}_n(X)|/n$ of a shift space X is a measure of the complexity of X. While the topological entropy is not computable for general subshifts, it is for sofic shifts (see [15]).

The language of a sofic shift is denoted as *regular* in the context of *formal language theory* (see [9] for an introduction). The class of regular languages is the class of languages which can be recognized by a *deterministic finite state automaton* (DFA). Formally, a DFA is a 5-tuple (Q, A, δ, q_0, F) where Q is a finite set of states, $F \subseteq Q$ is the set of *accepting* states, $q_0 \in Q$ is the *initial state*, A is a finite alphabet and $\delta : Q \times A \to Q$ is a partial transition function (i.e. it can be defined only on a subset of $Q \times A$). The language represented by a DFA is the set of words generated by following a path starting from the initial state and ending to an accepting state.

For every regular language there exists an unique smallest DFA, where smallest refers to the number of states. In general, most of the questions concerning regular languages are algorithmically decidable. In particular, it is decidable if two distinct DFA represent the same language.

From a DFA representing the language of a sofic shift S it is possible to derive a labeled graph presentation of S in the following way:

1. the set of vertices V consists of the pairs $(q, a) \in Q \times A$ s.t. $\delta(q, a) \in Q$.
2. there exists an edge $(q, a) \to (q', a')$, $(q, a), (q', a') \in V$, if $\delta(q, a) = q'$
3. $\forall v = (q, a) \in V$, $\zeta(v) = a$.

2.2 Cellular Automata

A *cellular automaton* is a dynamical system $(A^{\mathbb{Z}}, F)$ where A is a finite alphabet and F is a σ-commuting, continuous function. $(A^{\mathbb{Z}}, F)$ is generally identified by a block mapping $f : A^{2r+1} \to A$ such that $F(x)_i = f(x_{[i-r,i+r]}), i \in \mathbb{Z}$. According to Curtis-Hedlund-Lyndon Theorem [7], the whole class of continuous and σ-commuting functions between shift spaces arises in this way.

We refer to f and r respectively as *local rule* and *radius* of the CA.

A CA is *one-sided*, if the local rule is of the form $f : A^{r+1} \to A$ where $\forall x \in A^{\mathbb{Z}}, i \in \mathbb{Z}, F(x)_i = f(x_{[i,i+r]})$. A one-sided CA is usually denoted with $(A^{\mathbb{N}}, F)$.

We recall the definition of some topological properties of CA. Let d denote the metric on $A^{\mathbb{Z}}$ defined in Section 2.1.

Definition 1. *Let* $(A^{\mathbb{Z}}, F)$ *be a CA.*

1. $(A^{\mathbb{Z}}, F)$ *is* nilpotent *if*
$$\exists N > 0, \exists x \in A^{\mathbb{Z}}, \sigma(x) = x, \ s.t.\ \forall n \geq N, F^n(A^{\mathbb{Z}}) = x.$$
2. $(A^{\mathbb{Z}}, F)$ *is* equicontinuous at $x \in A^{\mathbb{Z}}$ *if*
$$\forall \epsilon > 0, \exists \delta > 0 \ s.t.\ \forall y \in A^{\mathbb{Z}}, d(x, y) < \delta, \exists n > 0 \ s.t.\ d(F^n(x), F^n(y)) < \epsilon.$$
3. $(A^{\mathbb{Z}}, F)$ *is* equicontinuous *if* $\forall x \in A^{\mathbb{Z}}$, $(A^{\mathbb{Z}}, F)$ *is equicontinuous at* x.

4. $(A^{\mathbb{Z}}, F)$ *is* almost equicontinuous *if* $\exists x \in A^{\mathbb{Z}}$ *s.t.* $(A^{\mathbb{Z}}, F)$ *is equicont. at* x.
5. $(A^{\mathbb{Z}}, F)$ *is* sensitive *if*

$$\exists \epsilon > 0 \text{ s.t. } \forall x \in A^{\mathbb{Z}}, \forall \delta > 0, \exists y \in A^{\mathbb{Z}}, d(x,y) < \delta, \exists n > 0 \text{ s.t. } d(F^n(x), F^n(y)) \geq \epsilon.$$

5. $(A^{\mathbb{Z}}, F)$ *is* positively expansive *if*

$$\exists \epsilon > 0 \text{ s.t. } \forall x, y \in A^{\mathbb{Z}}, x \neq y, \exists n > 0 \text{ s.t. } d(F^n(x), F^n(y)) \geq \epsilon.$$

Kari showed that nilpotency is an undecidable property [11]. In [4], Durand et al. showed that equicontinuity, almost equicontinuity and sensitivity are undecidable properties. Actually, it is unknown if positively expansiveness is or not a decidable property.

Definition 2. *(Column subshift) Let* $(A^{\mathbb{Z}}, F)$ *be a CA. For* $k > 0$ *let*

$$\Sigma_k = \{x \in (A^k)^{\mathbb{N}} \mid \exists y \in A^{\mathbb{Z}} : F^i(y)_{[0,k)} = x_i, i \in \mathbb{N}\}$$

denote the column subshift *of width* k *associated to* $(A^{\mathbb{Z}}, F)$.

Gilman noticed that the language of a column subshift is always context-sensitive [6]. Kůrka classified cellular automata according to the complexity of column subshifts languages [13].

Definition 3. *(Bounded periodic CA)* $(A^{\mathbb{Z}}, F)$ *is* bounded periodic *if* $\forall t > 0$, $\mathcal{L}(\Sigma_t)$ *is a bounded periodic language.*

Definition 4. *(Regular CA)* $(A^{\mathbb{Z}}, F)$ *is* regular *if* $\forall t > 0$, $\mathcal{L}(\Sigma_t)$ *is a regular language (or, equivalently, if* Σ_t *is sofic shift).*

Definition 5. *(Kůrka's Language classification) Every cellular automaton falls exactly in one of the following classes.*
L1. *Bounded periodic.*
L2. *Regular not bounded periodic.*
L3. *Not regular.*

Class $L1$ coincide with the class of equicontinuous CA [13]. Thus the membership in $L1$ is undecidable while it was unknown if it is for $L2, L3$.

The topological entropy $H(F) = \lim_{k\to\infty} h(\Sigma_k)$ of $(A^{\mathbb{Z}}, F)$ is a measure of the complexity of the dynamics of $(A^{\mathbb{Z}}, F)$. The problem of computing or even approximating the topological entropy of CA has been shown to be in general not algorithmically computable [8]. The topological entropy of one-sided CA has a simpler characterization than the general case (see [2]).

Theorem 1. *Let* $(A^{\mathbb{N}}, F)$ *be a CA with radius* r. *Then* $H(F) = h(\Sigma_r)$.

3 Results

In this section we investigate decidable properties of regular CA. Most of our effort will be devoted to show that if $S \subseteq (A^{2r+1})^{\mathbb{N}}$ is a sofic shift and $(A^{\mathbb{Z}}, F)$ is a CA with radius r, it is possible to decide whether $S = \Sigma_{2r+1}$ (Theorem 3). This strong result has a lot of consequences. The most relevant one is that for regular CA it is possible to compute column subshifts of every given width (Theorem 4). The (dynamical) complexity of a CA is strictly related to the complexity of column subshifts languages. Actually we show that, thanks to the computability property, it is possible to decide if a regular CA is nilpotent, equicontinuous or positively expansive (Theorem 6). Moreover, it turns also out, that it is possible to compute the topological entropy for one-sided regular CA (Theorem 5). The negative consequence of computability/decidability results is that regularity itself is an undecidable property (Theorem 7).

In order to show our fundamental decidability result (Theorem 3) we need to define the concept of *cellular automaton extension* of a sofic shift and to show some basic properties.

Definition 6. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r. Let $\mathcal{G} = (V, E, \zeta)$ be a labeled graph with $\zeta : V \to A^{2r+1}$. For $t > 0$, let the* (F,t)-extension *of $\mathcal{G}$ be the labeled graph $\mathcal{G}_{(F,t)} = (V_t, E_t, \zeta_t)$, with $\zeta_t : V_t \to A^{2r+t}$, defined in the following way (see figure 1):*

- vertex set:

$$V_t = \{(v_1, .., v_t) \in V^t \mid \exists a \in A^{2r+t}, \zeta(v_i) = a_{[i,2r+i]}, 1 \le i \le t\}$$

- edge set:

$$E_t = \{(e_1, .., e_t) \in E^t \mid \exists v, v' \in V_t, i(e_j) = v_j, t(e_j) = v'_j, f(\zeta(v_j)) = \zeta(v'_j)_{r+1}\}$$

- labeling function:

$$\forall v = (v_1, ..., v_t) \in V_t, \zeta_t(v) = a \text{ where } a_{[i,2r+i]} = \zeta(v_i), 1 \le i \le t.$$

Definition 7. *Let $(A^{\mathbb{Z}}, F)$ be a CA. Let $t > 0, k > 1$ and let $a, b \in \mathcal{B}_t(\Sigma_k)$ such that $a = a_1...a_k$, $b = b_1...b_k$ where $a_i, b_i \in A^t$ and $a_{i+1} = b_i, 1 \le i < k$. Then, we say that x, y are* compatible blocks *and we denote with $a \odot b = a_1...a_k b_k$ their* overlapping concatenation*.*

Moreover, let $x, y \in \Sigma_k$ such that $x = x_1..x_k, y = y_1...y_k$ where $x_i, y_i \in A^{\mathbb{N}}$ and $x_{i+1} = y_i, 1 \le i < k$. We say that x, y are compatible sequences *and, abusing the notation, we denote with $x \odot y = x_1...x_k y_k$ their* overlapping concatenation*.*

The following two lemmas will be used extensively.

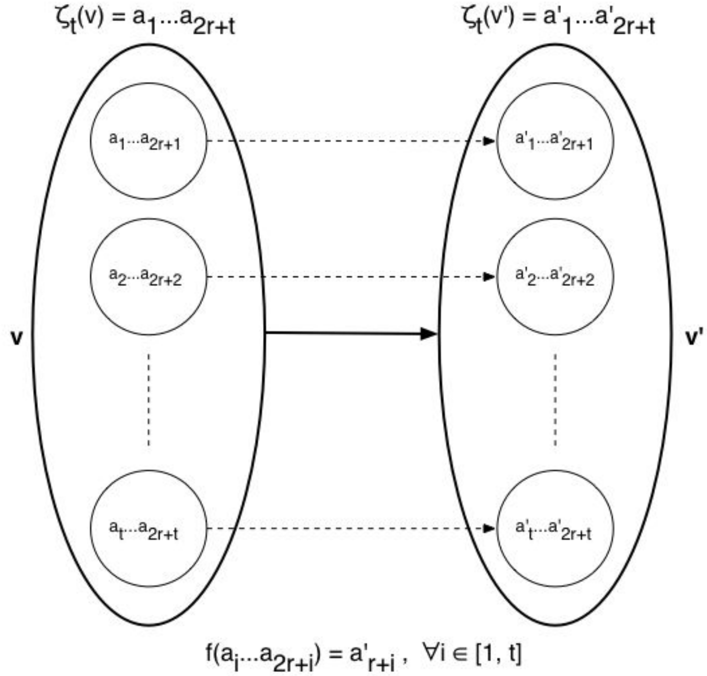

Fig. 1. A legal edge $v \to v'$ of an (F,t)-extended graph $\mathcal{G}_{(F,t)}$.

Lemma 1. *Let* $(A^{\mathbb{Z}}, F)$ *be a CA with radius* r*. Let* $t > 0$ *and let* $a, b \in \mathcal{B}_t(\Sigma_{2r+1})$ *be compatible blocks. Then* $a \odot b \in \mathcal{B}_t(\Sigma_{2r+2})$*.*

Proof. Let $a = a_1...a_t$ where $a_1, ..., a_t \in A^{2r+1}$ and let $x \in A^{\mathbb{Z}}$ such that $F^i(x)_{[0,2r]} = a_{i+1}$, $0 \le i < t$. Moreover, let $b = b_1...b_t$ where $b_1, ..., b_t \in A^{2r+1}$ and let $y \in A^{\mathbb{Z}}$ such that $F^i(y)_{[1,2r+1]} = b_{i+1}$, $0 \le i < t$. Let $z \in A^{\mathbb{Z}}$ be such that $z_{(-\infty,2r]} = x_{(-\infty,2r]}$, $z_{[1,\infty)} = y_{[1,\infty)}$ and let $a \odot b = c_1...c_t$ where $c_1, ..., c_t \in A^{2r+2}$. Then it is easy to check that $F^i(z)_{[0,2r+1]} = c_{i+1}$, $0 \le i < t$ which implies that $a \odot b \in B_t(\Sigma_{2r+2})$. □

Lemma 2. *Let* $(A^{\mathbb{Z}}, F)$ *be a CA with radius* r*. Let* $S \subseteq (A^{2r+1})^{\mathbb{N}}$ *be a sofic shift and let* $\mathcal{G}$ *be a labeled graph presentation of* S*. Let* $x, y \in S_{\mathcal{G}_{(F,1)}}$ *be compatible sequences. Then* $x \odot y \in S_{\mathcal{G}_{(F,2)}}$*.*

Proof. Since, by hypothesis, $x = (x_i)_{i\in\mathbb{N}}, y = (y_i)_{i\in\mathbb{N}} \in S_{\mathcal{G}_{(F,1)}}$, there exist two paths $u_1 \to u_2 \to ...$ and $v_1 \to v_2 \to ...$ in $\mathcal{G}$ such that $\zeta(u_i) = x_i$ and $\zeta(v_i) = y_i$, $i \in \mathbb{N}$. Then, $(u_1, v_1) \to (u_2, v_2) \to ...$ is a legal path in $\mathcal{G}_{(F,2)}$ which implies that $x \odot y \in S_{\mathcal{G}_{(F,2)}}$. □

The following proposition shows that the sofic shift presented by the (F,t)-extension $\mathcal{G}_{(F,t)}$ of a labeled graph $\mathcal{G}$ doesn't depend on $\mathcal{G}$ but only on the sofic shift presented by $\mathcal{G}$.

Proposition 1. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $\mathcal{G}, \mathcal{G}'$ be two distinct labeled graph presentations of the same sofic shift $S = S_{\mathcal{G}} = S_{\mathcal{G}'} \subseteq (A^{2r+1})^{\mathbb{N}}$. Then, for any $t > 0$, $S_{\mathcal{G}_{(F,t)}} = S_{\mathcal{G}'_{(F,t)}}$.*

Proof. We show that $S_{\mathcal{G}_{(F,t)}} \subseteq S_{\mathcal{G}'_{(F,t)}}$. The proof for the converse inclusion can be obtained by exchanging $\mathcal{G}$ with $\mathcal{G}'$.

First of all, note that, by definition of $(F,1)$-extension, $S_{\mathcal{G}_{(F,1)}} = S_{\mathcal{G}'_{(F,1)}}$. Let $x \in S_{\mathcal{G}_{(F,t)}}$ and let $x_1, ..., x_t \in S$ such that $x = x_1 \odot ... \odot x_t$. Then, $x_1, ..., x_t \in S_{\mathcal{G}'_{(F,1)}}$ and, by Lemma 2, it follows that $x \in S_{\mathcal{G}'_{(F,t)}}$. $\square$

Thanks to Proposition 1 we can refer directly to the extension of a sofic shift S rather than to the extension of a labeled graph presentation of S.

Definition 8. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r. Let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift and let $\mathcal{G}$ be a labeled graph presentation of S. For $t > 0$, let denote with $S_{(F,t)} = S_{\mathcal{G}_{(F,t)}}$ the* (F,t)-extension *of the sofic shift S.*

We now show some useful properties of the (F,t)-extensions of sofic shifts.

Lemma 3. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r. Let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. Then $\forall t > 0$,*

a. *if $\Sigma_{2r+1} \subset S$ then $\Sigma_{2r+t} \subseteq S_{(F,t)}$,*
b. *if $\Sigma_{2r+1} = S$ then $\Sigma_{2r+t} = S_{(F,t)}$,*
c. *if $\Sigma_{2r+1} \supset S$ then $\Sigma_{2r+t} \supset S_{(F,t)}$.*

Proof. **a.** Let $x \in \Sigma_{2r+t}$ such that $x = x_1 \odot .. \odot x_t$ where $x_i \in \Sigma_{2r+1}$, $1 \leq i \leq t$. Then, $x_i \in S_{(F,1)}$, $1 \leq i \leq t$ and, by Lemma 2, $x_1 \odot .. \odot x_t \in S_{(F,t)}$.

b. By point a, $\Sigma_{2r+t} \subseteq S_{(F,t)}$, thus we just have to show that $S_{(F,t)} \subseteq \Sigma_{2r+t}$ or, equivalently, that $\mathcal{L}(S_{(F,t)}) \subseteq \mathcal{L}(\Sigma_{2r+t})$. Let $k > 0$ and let $a \in \mathcal{B}_k(S_{(F,t)})$. Let $a_1, ..., a_t \in \mathcal{B}_k(S)$ be such that $a_1 \odot ... \odot a_t = a$. By hypothesis, $a_1, ..., a_t \in \mathcal{B}_k(\Sigma_{2r+1})$ then, by Lemma 1, it follows that $a_1 \odot ... \odot a_t \in \mathcal{B}_k(\Sigma_{2r+t})$.

c. Since $\Sigma_{2r+1} \supset S$, appling the same reasoning of point b, it is possible to conclude that $\Sigma_{2r+t} \supseteq S_{(F,t)}$. We have just to show that the inclusion is strict. Since $\Sigma_{2r+1} \supset S$, there exists a block $b_1 \in \mathcal{L}(\Sigma_{2r+1})$ such that $b_1 \notin \mathcal{L}(S)$. Then, let $b \in \mathcal{L}(\Sigma_{2r+t})$ such that $b = b_1 \odot b_2 \odot ... \odot b_t$ for some $b_2, ..., b_t \in \mathcal{L}(\Sigma_{2r+1})$. Trivially, $b \notin \mathcal{L}(S_{(F,t)})$. $\square$

The following theorem easily follows from Lemma 3 and provides a strong characterization for regular CA. It is a two-sided extension of a theorem proved by Blanchard and Maass for one-sided CA [1].

Theorem 2. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r. Then $(A^{\mathbb{Z}}, F)$ is regular if and only if Σ_{2r+1} is a sofic shift.*

Proof. The necessary implication is trivial. Then, suppose Σ_{2r+1} is a sofic shift. For every $d < 2r + 1$, Σ_d is a factor of Σ_{2r+1} then it is a sofic shift. For every $d > 2r+1$, by Lemma 3 point b, Σ_d can be represented by a labeled graph then it is a sofic shift. $\square$

In general, if Σ_d is a sofic shift for $d < 2r + 1$ it is not possible to conclude that the CA is regular (see [10]).

Definition 9. *Let A be a finite alphabet. Let $t \geq 1$ and let $[i,j] \subseteq [1,t]$ be an integer interval. Let*

$$\Phi_{[i,j]} : (A^t)^{\mathbb{N}} \to (A^{j-i+1})^{\mathbb{N}}$$

denote the projection map induced by the one-block factor map

$$\varphi_{[i,j]} : A^t \to A^{j-i+1}$$

defined by $\varphi_{[i,j]}(a_1...a_t) = a_i a_{i+1}...a_j, \forall a_1 a_2 ... a_t \in A^t$.

Remark 1. Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $\mathcal{G}_{(F,t)}$ be the (F,t)-extension of $\mathcal{G}$. Then for every $i \in [1,t]$, $\Phi_{[i,2r+i]}(S_{\mathcal{G}_{(F,t)}}) \subseteq S_{\mathcal{G}}$.

Definition 10. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. S is F-*extendible *if*

$$S = \Phi_{[i,2r+i]}(S_{(F,t)}), \forall t > 0, \forall i \in [1,t].$$

Note that for a sofic shift to be F-extendible is a necessary condition in order to be equal to Σ_{2r+1}.

Proposition 2. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. Then, S is F-extendible iff $S = \Phi_{[1,2r+1]}(S_{(F,2)}) = \Phi_{[2,2r+2]}(S_{(F,2)})$.*

Proof. The necessary implication is trivial. Then, let $S = \Phi_{[1,2r+1]}(S_{(F,2)}) = \Phi_{[2,2r+2]}(S_{(F,2)})$. Note that this implies $S = S_{(F,1)}$. Let $t > 2$, we have to show that $S = \Phi_{[i,2r+i]}(S_{(F,t)})$ for $1 \leq i \leq t$. Let $z \in S$ and let $k \in [1,t]$. To reach the proof it is sufficient to show that $z \in \Phi_{[k,2r+k]}(S_{(F,t)})$. Since $S = \Phi_{[1,2r+1]}(S_{(F,2)}) = \Phi_{[2,2r+2]}(S_{(F,2)})$, there exists $x_1, .., x_{t-1} \in S_{(F,2)}$ such that $\Phi_{[2,2r+2]}(x_i) = \Phi_{[1,2r+1]}(x_{i+1})$, $1 \leq i < t-1$ and $\Phi_{[2,2r+2]}(x_{k-1}) = \Phi_{[1,2r+1]}(x_k) = z$. Then, $x_1, .., x_{t-1}$ are compatible and by Lemma 2, it follows that $x_1 \odot ... \odot x_{t-1} \in S_{(F,t)}$ and $\Phi_{[k,2r+k]}(x_1 \odot ... \odot x_{t-1}) = z$. □

Proposition 3. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. Suppose S is F-extendible then $S \subseteq \Sigma_{2r+1}$.*

Proof. We prove by induction on $k > 0$ that $\mathcal{B}_k(S) \subseteq \mathcal{B}_k(\Sigma_{2r+1})$.

1. (Base Case) By definition, $\mathcal{B}_1(S) \subseteq \mathcal{B}_1(\Sigma_{2r+1}) = A^{2r+1}$.

2. (Inductive Case) Suppose $\mathcal{B}_k(S) \subseteq \mathcal{B}_k(\Sigma_{2r+1})$ for $k > 0$. We have to show that $\mathcal{B}_{k+1}(S) \subseteq \mathcal{B}_{k+1}(\Sigma_{2r+1})$.
Since the radius of the CA is r, the set of blocks $\mathcal{B}_{k+1}(\Sigma_{2r+1})$ is completely determined by the set of blocks $\mathcal{B}_k(\Sigma_{4r+1})$ as well as the set of blocks $\mathcal{B}_{k+1}(\Phi_{[r+1,3r+1]}(S_{(F,2r+1)}))$ is completely determined by the set of blocks $\mathcal{B}_k(S_{(F,2r+1)})$. Thus, showing that $\mathcal{B}_k(S_{(F,2r+1)}) \subseteq \mathcal{B}_k(\Sigma_{4r+1})$ we can reach the conclusion $\mathcal{B}_{k+1}(S) \subseteq \mathcal{B}_{k+1}(\Sigma_{2r+1})$.
Let $x \in \mathcal{B}_k(S_{(F,2r+1)})$. Since S is F-extendible, there exist $x_1, .., x_{2r+1} \in \mathcal{B}_k(S)$ such that $x = x_1 \odot ... \odot x_{2r+1}$. By inductive hypothesis, $x_1, ..., x_{2r+1} \in \mathcal{B}_k(\Sigma_{2r+1})$ then, by Lemma 1, $x \in \mathcal{B}_k(\Sigma_{4r+1})$. □

Proposition 4. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. Then it is decidable if S is F-extendible.*

Proof. Given a labeled graph representation of S, it is possible to compute $S_{(F,2)}$ and it is possible to compute labeled graph representations for $\Phi_{[1,2r+1]}(S_{(F,2)})$ and $\Phi_{[2,2r+2]}(S_{(F,2)})$. Given labeled graph representation of S, $S' = \Phi_{[1,2r+1]}(S_{(F,2)})$ and $S'' = \Phi_{[2,2r+2]}(S_{(F,2)})$ it is easy to build three finite state automata whose recognized languages are respectively $\mathcal{L}(S), \mathcal{L}(S')$ and $\mathcal{L}(S'')$. Then, the proof follows from Proposition 2 and from the decidability of the equivalence between finite state automata. □

Proposition 5. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq \Sigma_{2r+1}$ be a sofic shift. Then it is decidable if $\Sigma_{2r+1} = S$.*

Proof. We provide a proof for the following claim which trivially is algorithmically checkable.

> *Let $M = (Q, A^{2r+1}, q_0, F, \delta)$ be the smallest DFA recognizing the language $\mathcal{L}(S)$. Let $N = (|Q| \cdot |A|^{2r+1})^{2r+1}$. Then $\Sigma_{2r+1} = S$ if and only if $\mathcal{B}_N(\Sigma_{4r+1}) = \mathcal{B}_N(S_{(F,2r+1)})$.*

By Lemma 3, the necessary condition is trivially true. Obviously, if $\Sigma_{4r+1} = S_{(F,2r+1)}$ then $\Sigma_{2r+1} = S$. Thus, we show by induction on $k > 0$ that $\mathcal{B}_k(\Sigma_{4r+1}) = \mathcal{B}_k(S_{(F,2r+1)})$.

a. (Base Case) By hypothesis, $\mathcal{B}_N(\Sigma_{4r+1}) = \mathcal{B}_N(S_{(F,2r+1)})$. Moreover, since the language of a subshift is factorial, $\mathcal{B}_k(\Sigma_{4r+1}) = \mathcal{B}_k(S_{(F,2r+1)})$, $\forall k \leq N$.

b. (Inductive Case) Suppose $\mathcal{B}_K(\Sigma_{4r+1}) = \mathcal{B}_K(S_{(F,2r+1)})$, $K \geq N$. We have to show that $\mathcal{B}_{K+1}(\Sigma_{4r+1}) = \mathcal{B}_{K+1}(S_{(F,2r+1)})$.
Let $\mathcal{G} = (V, E, \zeta)$ be the labeled graph presentation of S derived from the smallest DFA M according to the procedure described at the end of section 2.1. Note that the number of vertices of $\mathcal{G}$ is less then or equal to $|Q| \cdot |A|^{2r+1}$. Moreover, let $\mathcal{G}_{(F,2r+1)}$ be the $(F, 2r+1)$-extension of $\mathcal{G}$. Note that the number of vertices of $\mathcal{G}_{(F,2r+1)}$ is less then or equal to N.
Let $a \in \mathcal{B}_{K+1}(\Sigma_{4r+1})$ and let $a^1, ..., a^{2r+1} \in \mathcal{B}_{K+1}(\Sigma_{2r+1})$ such that $a = a^1 \odot ... \odot a^{2r+1}$. Since, by inductive hypothesis, $\mathcal{B}_K(\Sigma_{4r+1}) = \mathcal{B}_K(S_{(F,2r+1)})$, it follows that $\mathcal{B}_{K+1}(\Sigma_{2r+1}) = \mathcal{B}_{K+1}(S)$ and, trivially, that $a^1, ..., a^{2r+1} \in \mathcal{B}_{K+1}(S)$. Then there exist uniques legal paths

$$u_1^1 \to ... \to u_{K+1}^1, \ ..., \ u_1^{2r+1} \to ... \to u_{K+1}^{2r+1}$$

in $\mathcal{G}$, where $u_1^i = (q_0, a_1^i)$ and $\zeta(u_k^i) = a_k^i, \forall i \in [1, 2r+1]$, $1 \leq k \leq K+1$.
We show that there exists $x \in S_{(F,2r+1)}$ such that $x_{[0,K]} = a$. Let $y \in S_{(F,2r+1)}$ such that $y_{[0,K-1]} = a_{[0,K-1]}$. One such y exists since, by inductive hypothesis, $\mathcal{B}_K(\Sigma_{4r+1}) = \mathcal{B}_K(S_{(F,2r+1)})$. Then there exists an unique path $v_0 \to v_1 \to ..$ in $\mathcal{G}_{(F,2r+1)}$ such that $\zeta_{2r+1}(v_i) = y_i$, $i \in \mathbb{N}$ and such that $v_0 = ((q_0, c_{[1,2r+1]}), ..., (q_0, c_{[2r+1,4r+1]}))$ where $c = y_0 \in A^{4r+1}$. Since $K > N$ there exist $0 \leq i < j < K$ such that $v_i = v_j$. Then,

let consider $\overline{a}^k = a_1^k a_2^k .. a_i^k a_{j+1}^k a_{j+2}^k ... a_{K+1}^k$, $1 \leq k \leq 2r+1$. Obviously, $\overline{a}^k \in \mathcal{L}(S) \cap \mathcal{L}(\Sigma_{2r+1})$, $1 \leq k \leq 2r+1$. Moreover, $\overline{a}^k$ are compatible then, by Lemma 1, $\overline{a} = \overline{a}^1 \odot ... \odot \overline{a}^{2r+1} \in \mathcal{L}(\Sigma_{4r+1})$ and, by inductive hypothesis, $\overline{a} \in \mathcal{L}(S_{(F,2r+1)})$.
Let $l = |\overline{a}|$. Let $z \in S_{(F,2r+1)}$ such that $z_{[0,l)} = \overline{a}$. Then, there exists an unique path $v'_0 \to v'_1 \to ..$ in $\mathcal{G}_{(F,2r+1)}$ such that $\zeta_{2r+1}(v'_i) = z_i$, $i \in \mathbb{N}$ and such that $v'_0 = v_0$. Moreover, since $v'_0 = v_0$ and $z_{[0,l)} = \overline{a}$, it follows that $v'_k = v_k$ for $0 \leq k \leq i$ and $v'_{i+k} = v_{j+k}$, $1 \leq k \leq C$ where $C = K - j$. Then it is easy to see that

$$v_0 \to ... \to v_K \to v'_{i+C+1} \to v'_{i+C+2} \to ...$$

is a legal path in $\mathcal{G}(F, 2r+1)$ and that the labelings of the vertices in the path generate a sequence $x \in S_{(F,2r+1)}$ such that $x_{[0,K]} = a$. □

Now we are ready to state our main result and next to show the most immediate consequences.

Theorem 3. *Let $(A^{\mathbb{Z}}, F)$ be a CA with radius r and let $S \subseteq (A^{2r+1})^{\mathbb{N}}$ be a sofic shift. Then it is decidable if $S = \Sigma_{2r+1}$.*

Proof. $S = \Sigma_{2r+1}$ if and only if S is F-extendible and $S \supseteq \Sigma_{2r+1}$. Then, the proof follows from Proposition 4 and Proposition 5. □

We now explore some important consequences of Theorem 3 related to regular CA.

Theorem 4. *Let $(A^{\mathbb{Z}}, F)$ be a regular CA. Then $\forall t > 0$, Σ_t is computable.*

Proof. Let r be the radius of the CA. By Theorem 3, given a sofic shift $S \subseteq (A^{2r+1})^{\mathbb{N}}$, it is possible to decide if $S = \Sigma_{2r+1}$. We can enumerate all labeled graph representing all sofic shifts contained in A^{2r+1}. Then there exists an algorithm that iteratively generates graphs in the enumeration and checks if the shift represented is Σ_{2r+1}. Since $(A^{\mathbb{Z}}, F)$ is regular, Σ_{2r+1} will be eventually generated and recognized. This proves that, if $(A^{\mathbb{Z}}, F)$ is regular, Σ_{2r+1} is computable.

In general, if $t < 2r+1$, we can compute Σ_t by simply taking the projection $\Phi_{[1,t]}(\Sigma_{2r+1})$ otherwise, if $t > 2r+1$, by Lemma 3 point b, we can compute Σ_t by computing the $(F, t-2r)$-extension of Σ_{2r+1}. □

The following theorem gives an answer to a question raised in [3].

Theorem 5. *The topological entropy of one-sided regular CA is computable.*

Proof. Since the entropy of sofic shifts is computable, the conclusion follows from Theorem 1 and Theorem 4. □

The following theorem shows that if we restrict to the class of regular CA, it is possible to provide answers to questions which are undecidable in the general case.

Theorem 6. *Let $(A^{\mathbb{Z}}, F)$ be a regular CA. Then the following topological properties are decidable.*

1. *Nilpotency*
2. *Equicontinuity*
3. *Positively Expansiveness*

Proof. By Theorem 4, given $(A^{\mathbb{Z}}, F)$, it is possible to compute Σ_{2r+1}.

1. It is easy to see that $(A^{\mathbb{Z}}, F)$ is nilpotent if and only if there exists $a \in A^{2r+1}$ and $N > 0$ such that $\forall n \geq N, \forall x \in \Sigma_{2r+1}$, $\sigma^n(x) = a$. Given a labeled graph representation of Σ_{2r+1}, this last condition is trivially algorithmically checkable.
2. It is easy to see that $(A^{\mathbb{Z}}, F)$ is equicontinuous if and only if $\mathcal{L}(\Sigma_{2r+1})$ is a bounded periodic language and that, given a labeled graph representation of Σ_{2r+1}, it is algorithmically checkable if $\mathcal{L}(\Sigma_{2r+1})$ is bounded periodic.
3. Every positively expansive CA is conjugated to (Σ_{2r+1}, σ) where Σ_{2r+1} is a shift of finite type and, in particular, it is an n-full shift (see [12]). Since, for positively expansive CA, $n = |F^{-1}(x)|$ for every $x \in A^{\mathbb{Z}}$, n is a computable number. The proof follows from the decidability of the conjugacy problem for one-sided shifts of finite type (see [15]). □

To conclude, we show that, as a negative consequence of the decidability of properties in Theorem 6, regularity is an undecidable property which implies that the membership in Kůrka's language classes is undecidable.

Theorem 7. *It is undecidable whether a CA is regular.*

Proof. Assume it is decidable if a CA is regular. Then, since nilpotent CA are regular, by Theorem 6, it is possible to decide if a CA is nilpotent. □

4 Conclusions and open problems

We investigated decidable properties for regular cellular automata. We showed that regularity itself is not a decidable property (Theorem 7) and that, conversely, for regular cellular automata nilpotency, equicontinuity and positively expansiveness are decidable properties (Theorem 6). Moreover we aswered a question raised in [3] showing that the topological entropy is computable for one-sided regular CA (Theorem 5). It is unknown if almost equicontinuity and sensitivity are or not decidable properties for regular CA (since to be almost equicontinuous or sensitive is a dicotomy for CA, this two properties are either both decidable or both not decidable).

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