THE POSSIO INTEGRAL EQUATION OF AEROELASTICITY: A MODERN VIEW

A.V. Balakrishnan[1]
[1] *University of California, Department of Electrical Engineering, Los Angeles , CA, USA*
*bal@ee.ucla.edu**

Abstract A central problem of AeroElasticity is the determination of the speed of the aircraft corresponding to the onset of an endemic instability known as wing 'flutter'. Currently all the effort is completely computational:wedding Lagrangian NASTRAN codes to the CFD codes to produce 'Time Marching' solutions. While they have the ability to handle non-linear complex geometry structures as well as viscous flow,they are based approximation of the p.d.e. by o.d.e., and restricted to specified numerical parameters.This limits generality of results and provides little insight into phenomena. And of course are inadequate for Control Design for stabilization. Retaining the continuum models,we can show that the basic problem is a Boundary Value/Control problem for a pair of coupled partial differential equations,and the composite problem can be cast as a nonlinear Convolution/Evolution equation in a Hilbert Space. The Flutter speed can then be characterized as Hopf Bifurcation point, and determined completely by the linearised equations. Solving the linearised equations is equivalent to solving a singular integral equation discovered by Possio in 1938 for oscillatory response.In this paper we examine the Equation and its generalizations from the modern mathematical control theory viewpoint.

keywords: Possio Equation, AeroElasticity, Instability, Wing Flutter

1. Introduction

The genesis of the Possio Equation and its role in the Aeroelasticity theory of the 1950's has been amply documented in [1]. This paper presents the current outlook on this equation, including generalizations, from the vantage point of recently developed control theory for partial differential equations [2].

A central problem of AeroElasticity is the stability of the wing structure in air flow. Much of the interest is in subsonic compressible flow. This can be formulated (see[3]) in the Time Domain as a nonlinear convolution/evolution equation in a Hilbert Space,and the instability ('Flutter') speed as a Hopf Bi-

*Research supported in part under NSF Grant No. ECS-040073.

Please use the following format when citing this chapter:
Author(s) [insert Last name, First-name initial(s)], 2006, in IFIP International Federation for Information Processing, Volume 199, System Modeling and Optimization, eds. Ceragioli F., Dontchev A., Furuta H., Marti K., Pandolfi L., (Boston: Springer), pp. [insert page numbers].

furcation point which by the Hopf theory is completely determined by the linearised equations about the undisturbed flow. The linearised equations are of the Neumann boundary type, and hence can be cast equivalently as an Integral Equation – this is the Possio Equation,with a singular kernel.

We may also place it in the context of the currently fashionable numerical computation schemes-indeed almost all the work in AeroElasticity today is computational. In essence the partial differential equations are approximated by ordinary differential equations – both Structural Dynamics and AeroDynamics,and the most subjective part-often mysterious even-is the wedding of the Lagrangian Structure Dynamics to the Eulerian AeroDynamics. This is exactly where the Possio Equation would come in,if the full continuum models are retained.

We begin in section 2 with the Wing Structure model,where we need to calculate the aerodynamic loading In section 3 we consider the AeroDynamic flow model-the Euler Full Potential Equation with aeroelastic boundary conditions for attached flow, and the Kutta-Joukowsky conditions. The linearization of the equations is in section 4. Finally in section 5 we study the role of the Possio Equation.

2. Structure Model

The simplest model – a uniform rectangular beam, endowed with two degrees of freedom, plunge and pitch – goes back to Goland [4] in 1945 (too late for Possio!). Let the projection of the flow velocity be along the positive $X-$ axis,with x denoting the chord variable, $-b \leq x \leq b$. Similarly with y denoting the span or length variable, along the $Y-$ axis, $0 \leq y \leq l$, let

$$X(y,t) = \text{Column}\,(h(y,t), \theta(y,t)),$$

where $h(\cdot)$ is the plunge or bending along Z-axis; and $\theta(\cdot)$ is the pitch or torsion angle about the elastic axis located at $x = ab$. Then the structure dynamics equation is:

$$M_S\,\ddot{X}(.,t) + KX(.,t) = \text{Column}\,(L(.,t), M(.,t)),$$

where M_S is the Mass/Inertia matrix and K is the stiffness differential operator:

$$\text{Diagonal}\,(EI\frac{\partial^4}{\partial y^4},\ -GJ\frac{\partial^2}{\partial y^2}),$$

$L(\cdot)$ is the aerodynamic lift and $M(\cdot)$ the moment about the elastic axis, with boundary conditions: a) Cantilever

$$h(0,t) = h'(0,t) = 0 = \theta(0,t) = \theta'(l,t) = h'''(l,t) = 0 = h''(l,t).$$

b) Free-Free

$$\theta'(0,t) = \theta'(l,t) = h'''(0,t) = h'''(l,t) = h''(0,t) = h''(l,t) = 0.$$

See [5,6] for a Hilbert space formulation. The functions $L(\cdot)$ and $M(\cdot)$ have to be determined from the Aerodynamic model.

3. AeroDynamic Model

The aerodynamic lift and moment (per unit length) are given by:

$$L(y,t) = \int_{-b}^{b} \delta p \, dx$$
$$M(y,t) = \int_{-b}^{b} (x-a)\delta p \, dx$$
$$\delta p = p(x,y,0+,t) - p(x,y,0-,t), 0 < y < l; |x| < b$$

where $p(x,y,z,t)$ is the aerodynamic pressure,which along with the velocity vector $q(x,y,z,t)$, the density $\varrho(x,y,z,t)$ are the basic aerodynamic variables of interest. Under some simplifying assumptions (see[8]), we can show that the velocity is curl-free and is then characterized by the velocity potential $\phi(x,y,z,t)$ which satisfies the (Euler) Full Potential Equation:

$$\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial(\nabla\phi.\nabla\phi)}{\partial t} = a_\infty^2 \nabla.\nabla\phi(1 + \frac{(\gamma-1)}{a_\infty^2}(\frac{(q_\infty.q_\infty)}{2} - \frac{\partial\phi}{\partial t}$$
$$- \frac{(\nabla\phi).(\nabla\phi)}{2})) - \nabla\phi.\nabla(\frac{\nabla\phi.\nabla\phi}{2})$$

where q_∞ is the undisturbed far-field velocity, a_∞ is the far field speed of sound, ϱ_∞ is the far field density.

$$M(\text{Mach Number}) = (\frac{|q_\infty|}{a_\infty}) \leq 1.$$

The pressure is given by

$$p = \varrho_\infty^2 \frac{a_\infty^2}{\gamma} \left(1 + (\frac{(\gamma-1)}{a_\infty^2})(\frac{(q_\infty.q_\infty)}{2} - \frac{\partial\phi}{\partial t} - \frac{(\nabla\phi.\nabla\phi)}{2})\right)^{\gamma/(\gamma-1)}$$

It is assumed that the far field potential is given by

$$\phi_\infty = q_1\, x + q_2\, y + q_3\, z$$

where

$$q_i = |q_\infty| \cos\alpha_i$$

AeroElastic Boundary Conditions.

The aeroelastic boundary conditions are:

a) Attached Flow

$$\frac{\partial \phi}{\partial z}|_{z=0} = \frac{\partial \phi_\infty}{\partial z}|_{z=0} + w_a(x, y, t)$$

where $w_a(\cdot)$ is the normal velocity of the structure, and is given by:

$$\begin{aligned} w_a(x, y, t) &= -\dot{h}(y, t) - (x - a\,b)\,\dot{\theta}(y, t)) - \frac{\partial \phi(x, y, 0, t)}{\partial x}\,\theta(y, t) \\ &\quad -\frac{\partial \phi(x, y, 0, t)}{\partial y}\,(h'(y, t) + (x - a)\theta'(y, t)). \end{aligned}$$

b) Kutta-Joukowsky Condition:

$\delta p = 0$, off the structure and at the trailing edge (goes to 0, as $x \to b_-$),

where δp is the pressure jump :

$$\delta p = p(x, y, 0+, t) - p(x, y, 0-, t)$$

We do not have an existence theorem for this problem as yet!

4. Linearization

Because of the lack of existence theorem and other reasons it is customary to simplify the Full Potential Equation to the Transonic Small Disturbance Potential (TSD) equation which is quasilinear and yet retains sufficient non-linearity to yield shocks – see [8]. Here however we go straight to the linearzation. Thus defining

$$\varphi = \phi - \phi_\infty$$

we have:

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} + 2U(q_1 \frac{\partial^2 \varphi}{\partial x \partial t} + q_2 \frac{\partial^2 \varphi}{\partial y \partial t} + q_3 \frac{\partial^2 \varphi}{\partial z \partial t}) &= a_\infty^2((1 - M^2 q_1^2) \frac{\partial^2 \varphi}{\partial x^2} \\ +(1 - M^2 q_2^2) \frac{\partial^2 \varphi}{\partial y^2} + (1 - M^2 q_3^2) \frac{\partial^2 \varphi}{\partial z^2}), & \end{aligned} \tag{1}$$

where now

$$U = |(q_\infty)|; \quad q_i = U\,\cos\alpha_i; \quad M = \frac{U}{a_\infty}$$

The boundary conditions are

$$\frac{\partial \varphi}{\partial z} = w_a(x, y, t), \quad 0 < y < l; \ |x| < b, \tag{2}$$

where

$$\begin{aligned} w_a(x,y,t) &= -\dot{h}(y,t) - (x - a\,b)\,\dot{\theta}(y,t)) - q_1\theta(y,t) \\ &\quad -q_2\,(h'(y,t) + (x - a\,b)\,\theta'(y,t)). \end{aligned}$$

With ψ denoting the linearised acceleration potential

$$\psi = \frac{\partial \varphi}{\partial t} + q_1 \frac{\partial \varphi}{\partial x} + q_2 \frac{\partial \varphi}{\partial y} + .q_3 \frac{\partial \varphi}{\partial z}$$

the Kutta-Jukowski conditions become:

$$\delta\,\psi = \psi|_{z=0+} - \psi|_{0-} = 0, \qquad \text{off the structure,} \tag{3}$$

$$\delta\,\psi \to 0 \text{ as } x \to b-, \quad 0 < y < l. \tag{4}$$

These are the 3-D linear subsonic Compressible flow conditions with the aeroelastic boundary conditions – see [8] for more details.

5. The Possio Integral Equation

Let us begin with a statement of the Possio Integral Equation — actually this is a generalization of the original equation bearing his name which was 2-D, zero angle of attack, Fourier Transform (sinusoidal response) version. We state it for the 3-D case, in terms of the Laplace Transform variable λ, Re $\lambda > \sigma \geq 0$, because the integrals defining the equation will be convergent (which is not the case for $\lambda = i\omega$, as in the original formulation). Let

$$\begin{aligned} \widehat{w}_a(x,y,\lambda) &= \int_0^\infty e^{-\lambda t}\, w_a(x,y,t)\,dt \\ A(x,y,t) &= -\frac{2}{U}\,\delta\,\psi \\ \widehat{A}(x,y,\lambda) &= \int_0^\infty e^{-\lambda t}\, A(x,y,t)\,dt \end{aligned}$$

To reduce complexity, we shall take

$$q_1 = 1 \text{ (zero angle of attack)}$$

see [8] for the case $0 < q_1 < 1$. Then the equation is (see [9])

$$\widehat{w}_a(x,y,\lambda) = \int_0^l \int_{-b}^b \widehat{P}(x-\zeta, y-\nu, k)\,\widehat{A}(\zeta,\nu,\lambda)d\zeta\,d\nu \tag{5}$$

where $|x| \leq b$, $0 \leq y \leq l$ and

$$k = \frac{\lambda}{U}$$

and the spatial Fourier Transform of the kernel $\widehat{P}(\cdot,\cdot,k)$ is

$$\begin{aligned}\widehat{\widehat{P}}(i\omega_1, i\omega_2, k) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\omega_1 x+\omega_2 y)}\widehat{P}(x,y,k)\,dx\,dy,\\ &= \frac{\sqrt{D(i\omega_1, i\omega_2, k)}}{2(k+i\omega_1)}, \quad -\infty<\omega_1,\omega_2<\infty, \qquad (6)\end{aligned}$$

where

$$D(i\omega_1, i\omega_2, k) = M^2k^2 + 2kM^2\, i\omega_1 + (1-M^2)\omega_1^2 + \omega_2^2. \qquad (7)$$

We prefer the succinct form of the spatial Fourier Transform in contrast to the $\widehat{P}(\cdot,\cdot,k)$ which is too long to specify see [10,13]. It has a singularity at the origin so that we have a singular integral equation [9]. Assume that (5) has a solution. Then the solution of the linearised potential equation (1) specialized for $q_1 = 1$,q_2,q_3 both zero , is given by:

$$\begin{aligned}\widehat{\widehat{\varphi}}(i\omega_1, i\omega_2, z, \lambda) &= \frac{\widehat{\widehat{A}}(i\omega_1, i\omega_2, \lambda)}{k+i\omega_1}\, e^{-z\sqrt{D(i\omega_1, i\omega_2, k)}},\ z>0,\\ &= -\widehat{\widehat{\varphi}}(i\omega_1, i\omega_2, -z, \lambda),\ \text{ for } z<0\end{aligned}$$

where $-\infty < \omega_1, \omega_2 < \infty$ and

$$\widehat{\widehat{\varphi}}(i\omega_1, i\omega_2, z, \lambda) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widehat{\varphi}(x,y,z,\lambda)\, e^{-(i\omega_1 x + i\omega_2 y)}\,dx\,dy,$$

$$\widehat{\varphi}(x,y,z,\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t}\,\varphi(x,y,z,t)\,dt,$$

$$\widehat{\widehat{A}}(i\omega_1, i\omega_2, \lambda) = \int_0^l\int_{-b}^{b} e^{-(i\omega_1 x + i\omega_2 y)}\widehat{A}(x,y,\lambda)\,dx\,dy.$$

This is essentially a formula due to Kussner, an early German pioneer (see [1]). We note that the existence of solution to (5) is still an open problem, despite early work on the problem [11].
par To obtain the original $2D$ version of Possio we need to specialize to the 'airfoil' case – or, 'high-aspect-ratio' wings where

$$\frac{l}{b} \approx \infty$$

so that we may neglect the dependence on the y-coordinate . With q_1 equal to unity, this becomes

$$\frac{\partial^2\varphi}{\partial t^2} + 2U\frac{\partial^2\varphi}{\partial x\partial t} = a_\infty^2((1-M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2}). \qquad (8)$$

And correspondingly (5) becomes:

$$\widehat{w}(x,y,\lambda) = \int_{-b}^{b} \widehat{P}(x-s,k)\widehat{A}(s,\lambda)\,ds, |x| \leq b \tag{9}$$

where setting ω_2 in (5) to be zero, we have, for $\omega \in (-\infty, \infty)$,

$$\widehat{\widehat{P}}(i\omega,k) = \int_{-\infty}^{\infty} \widehat{P(x,k)}e^{-i\omega x}dx = \frac{\sqrt{k^2M^2 + 2kM^2 i\omega + (1-M^2)\omega^2}}{2(k+i\omega)}, \tag{10}$$

where we have discarded the subscript 1.Ê In this case it becomes actually a Mikhlin multiplier – see [12].

Second we need to consider the case of 'oscillatory' response-Fourier Transform in the time-domain; formally putting $i\omega$Ê for λ everywhere. In this case, the corresponding kernel function becomes rather involved and too long to specify [13]; further, the integrals in the kernel function also require special interpretation as in [10].

The importance of the Possio equation is that it links directly the structure velocity-the "input' in the problem to the 'output' Ê– the pressure jump which is all that is needed in the aeroelastic problem. We do NOT need to solve for the potential everywhere. On the other hand the potential can be determined from the pressure jump – this is the formula of Kussner (8). Thus solving the Possio equation is equivalent to solving the boundary value problem for the potential. It is true that this holds only for the linearised equations-we don't have yet a 'non-linear' Possio Integral equation. But if stability – or Flutter speed – is the prime concern,then all we need is the solution to the Possio equation! Given this,it is surprising there is hardly a mention of this equation in recent Texts [15]. Indeed, a systematic use of the Possio equation would have reduced the size of the classic text [13]. Finally we note at present the existence/uniqueness of solutions to the Possio Equation is known only for the 'air-foil' case and even at that only for $M = 0$ and $M = 1$, (see [14]). See [7] for some approximations. Otherwise the problem is open.

6. References

[1] R. Voss. The Legacy of Camillo Possio to Unsteady AeroDynamics. Proceedings. This Conference.

[2] I. Lasiecka and R. Triggiani: Control Theory for Partial Differential Equations: Continuous and approximation theories. Cambridge University Press 2000.

[3] A.V. Balakrishnan. NonLinear AeroElasticity: Continuum Theory : Flutter/Divergence Speed: Plate Wing Model. Journal of AeroSpace Engineering,

2005, To appear.

[4] M. Goland: The flutter of a uniform cantilever wing. Journal of Applied Mechanics. Transactions ASME 12 : 197-208,1945.

[5] A.V. Balakrishnan. Subsonic Flutter Suppression using self-straining actuators. Journal of the Franklin Institute 338: 149-170,2001.

[6] A.V. Balakrishnan , M.A. Shubov: Asymptotic and spectral properties of operator valued functions generated by aircraft wing model. Math. Meth. Appl.Sci. 2004, 27:329-362.

[7] A.V. Balakrishnan,æ K.W. Iliff: A Continuum AeroElastic Model for Inviscid Subsonic Bending-Torsion Wing Flutter. In Proceedings of the International Forum on AeroElasticity and Structural Dynamics, Amsterdam, June 4-6, 2003, Amsterdam.

[8] A.V. Balakrishnan. On the Transonic Small Disturbance Potential Equation. AIAA Journal 42: 1081-1088, 2003.

[9] A.V. Balakrishnan. On the NonNumeric Mathematical Foundations of Linear AeroElasticity. 4th Int. Conf. on NonLinear Problems in Aviation & Aerospace, European Conference Publications, U.K., 2003.

[10] C.E. Watkins, H.L. Runyon, D.S. Woolston. . The kernel function of the Integral Equation relating the lift to the downwash distribution in oscillating finite wings in subsonic flow. NACA TN 1234, 1955.

[11] E. Reissner: On the theory of oscillating air foils of finite span in subsonic compressible flow. NACA TR 1002,1950.

[12] S.G. Mikhlin: MultiDimensional singular integrals and integral equations. Pergamon 1965.

[13] R.L. Bisplinghof, H. Ashley and R.L Halfman: AeroElasticty, Addison-Wesley, 1955.

[14] A.V. Balakrishnan: Possio Integral of AeroElasticity Theory. Journal of AeroSpace Engineering. 16: 139 -154.

[15] E. Dowell et al. A Modern Course in AeroElasticity. Kluwer 2004.