Stability Analysis of Imprecise Prey-Predator Model

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Abstract. Since the last few decades, the prey-predator system delivers attractive mathematical models to analyse the dynamics of prey-predator interaction. Due to the lack of precise information about the natural parameters, a significant number of research works have been carried out to take care of the impreciseness of the natural parameters in the prey-predator models. Due to direct impact of the imprecise parameters on the variables, the variables also become imprecise. In this paper, we developed an imprecise prey-predator model considering both prey and predator population as imprecise variables. Also, we have assumed the parameters of the prey-predator system as imprecise. The imprecise prey-predator model is converted to an equivalent crisp model using "e" and "g" operator method. The condition for local stability for the deterministic system is obtained mathematically by analysing the eigenvalues of the characteristic equation. Furthermore, numerical simulations are presented in tabular and graphical form to validate the theoretical results.

Keywords: Prey-Predator· Local Stability· Imprecise Environment· "e" and "g" operator method

1 Introduction

Mathematicians are provoked by the problems of understanding the biological phenomena. By quantitatively describing the biological problems, the mathematical researchers applied various mathematical tools to analyse and interpret the results. Mathematical areas as calculus, differential equations, dynamical systems, stability theory, fuzzy set theory etc. are being applied in biology. Biological phenomena like prey-predator interaction, prey-predator fishery harvesting system, the prey-predator system with infection, etc. can be expressed by an autonomous or non-autonomous system of ordinary differential equations.

2 Related Work

Work in the area of theoretical biology was first introduced by Thomas Malthus in the late eighteenth century, which later is known as the Malthusian growth model. The Lotka[8] and Volterra[16] predator-prey equations are other famous examples. Till then, a significant development in the area of population dynamics has been made by the researchers. Kar [6] formulated and analysed a prey-predator harvesting problem incorporating a prey refuge, Chakraborty et al. [1], solved stage-structured prey-predator harvesting models. Qu and Wei [13] have presented bifurcation analysis in a stage-structure prey–predator growth model. Seo and DeAngelis [14] formulated a predator-prey model with a Holling response function of type I and many more. From the above works, it can be observed that the biological parameters whichever were used in the models are always fixed, but in reality they vary under dynamical ecological conditions. Not only that, due to lack of precise numerical information such as experimental part, data collection, measurement process, determining initial condition, some parameters become imprecise. Also, in deterministic dynamical system parameters need to be precisely defined. To have a rough estimation of the parameters, a huge amount information is needed to continue processed. Imprecise bio-mathematical models are more meaningful than the deterministic models. There is a long history of imprecise prey-predator model. To mention a few, Guo et al. [5] established fuzzy impulsive functional differential equation using Hullermeiers approach of a population model. Peixoto et al. [2] presented the fuzzy predator-prey model. Pal et. al. [11], De et al. [3, 4] investigated optimal harvesting of fishery-poultry system with interval biological parameters. Stability Analysis of Predator-Prey System with Fuzzy Impulsive Control done by Wang [17]. Tapaswini and Chakraverty [15] numerically solved of Fuzzy Arbitrary Order Predator-Prey Equations. Stability and bionomic analysis of fuzzy parameter based prey-predator harvesting model using UFM also done by Pal et al. [10]. Due to direct effect of the imprecise parameters on the variables, the variables also become imprecise in nature. But, in most of the research work it is found that only the parameters or the coefficients involved in the models are assumed to be imprecise. Khatua and Maity [7] have analysed the stability of fuzzy dynamical systems based on a quasi-level-wise system where all the variables along with the parameters are considered as imprecise variables and using "e" and "g" operator method the imprecise model converted to an equivalent crisp problem.

From previous research, though we found a significant amount of research in the area of the prey-predator system in imprecise environment, to the best of our knowledge, none of the articles have introduced the impreciseness in the variables along with the parameters. In this work, 1) We have developed an imprecise prey-predator model. We have considered both prey and predator population as imprecise variables. 2) Also, we have assumed the parameters, namely growth rate of prey, predation rate of the prey population, increase rate and decay rate of the predator population as imprecise parameters. 3) Following Khatua et al. [7], the imprecise prey-predator model is converted to equivalent crisp model using "e" and "g" operator method. The local stability analysis is done for the deterministic system mathematically. The Numerical result is presented in tabular and graphical form to validate the theoretical findings.

The rest of the paper is arranged in the following manner:In section 3 Some mathematical preliminaries are mentioned. Section 4 is used for assumptions and notations. The prey-predator model is formulated in a crisp environment in section 5. In section 6, the model is transformed into an imprecise model, and after that, the imprecise model converted to equivalent crisp model using "e" and "g" operator method. Then the local stability of two different cases is analysed theoretically, numerically and presented graphically in this section. The results obtained in the numerical experiment are discussed in section 7. Finally, the chapter is concluded in section 8.

3 Mathematical Preliminaries

Mathematical preliminaries are recollected in this section.

3.1 Basic Concept of "e" and "g" Operators

Let $\mathbf{C}$ be a complex set *i.e* $\mathbf{C} = \{a + ib : a, b \in \Re\}$. Then "e" is a identity operator and "g" corresponds to a flip about the diagonal in the complex plane, *i.e.*, $\forall a + ib \in \mathbf{C}$,

$$\begin{cases} e : a + ib \to a + ib, \\ g : a + ib \to b + ia. \end{cases} \tag{1}$$

3.2 Use of "e" and "g" Operators to Fuzzy Dynamical System

Let us consider the following non-homogeneous fuzzy dynamical system

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t) + \mathbf{f}(t), \quad \tilde{\mathbf{x}}(0) = \tilde{\mathbf{x}}_0, \quad t \in [0, \infty) \tag{2}$$

where $\dot{\tilde{\mathbf{x}}}(t) = [\dot{\tilde{x}}_1(t) \cdots \dot{\tilde{x}}_n(t)]^T$ and $\mathbf{f}(t) = [f_1(t) \cdots f_n(t)]^T$.
Let $\overline{\mathbf{Y}}^\alpha(t) = [\bar{y}_1^\alpha(t) \cdots \bar{y}_n^\alpha(t)]^T$, $\underline{\mathbf{Y}}^\alpha(t) = [\underline{y}_1^\alpha(t) \cdots \underline{y}_n^\alpha(t)]^T$ be the solutions of

quasi-level-wise system

$$\begin{cases} \dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t) = B[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)), \\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha} \end{cases} \tag{3}$$

where $\underline{\mathbf{f}}(t) = \bar{\mathbf{f}}(t) = \mathbf{f}(t)$ and $\mathbf{B} = [b_{ij}]_{n\times n},\ b_{ij} = \begin{cases} a_{ij}e & a_{ij} \geq 0 \\ a_{ij}g & a_{ij} < 0 \end{cases}$. Then $\bar{x}_i^{\alpha}(t) = \max\limits_{t\in(0,\infty)} \{\bar{y}_i^{\alpha}(t), \underline{y}_i^{\alpha}(t)\}$
$\underline{x}_i^{\alpha}(t) = \min\limits_{t\in(0,\infty)} \{\bar{y}_i^{\alpha}(t), \underline{y}_i^{\alpha}(t)\}, i = 1, 2, \cdots, n$ are also the solutions of the fuzzy dynamical system (2).
Now if the dynamical system (3) is unstable, then moving the instability property to the level-wise system, we get to the same system as (3) i.e.,

$$\begin{cases} e[\dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t)] = \bar{\mathbf{B}}[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)), \\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha} \end{cases} \tag{4}$$

or

$$\begin{cases} g[\dot{\underline{\mathbf{Y}}}^{\alpha}(t) + i\dot{\overline{\mathbf{Y}}}^{\alpha}(t)] = \bar{\mathbf{B}}[\underline{\mathbf{Y}}^{\alpha}(t) + i\overline{\mathbf{Y}}^{\alpha}(t)] + (\underline{\mathbf{f}}(t) + i\bar{\mathbf{f}}(t)), \\ \underline{Y}^{\alpha}(0) = \underline{x}_0^{\alpha}, \quad \overline{Y}^{\alpha}(0) = \overline{x}_0^{\alpha} \end{cases} \tag{5}$$

where $\bar{\mathbf{B}} = [\bar{b}_{ij}]_{n\times n},\ \bar{b}_{ij} = a_{ij}e$ or $a_{ij}g$

4 Assumption & Notations

Here we have used the following notations: X, Y: the prey and predator population density at any time t respectively.
s: the natural growth rate of prey population.
K: the environmental carrying capacity at any time t.
δ: predation rate of prey population by prey population.
β: increase rate of predator population due to successful predation of prey.
γ: decay rate of predator population due to natural death.
We have assumed that the prey population do not have any decay due to natural death. The system is closed and there is no external harvesting of prey or predator.

5 Formulation of the model in crisp environment

The present study analyses a prey-predator model. The prey population follows a logistic growth model with s as intrinsic growth rate and K to be environmental carrying capacity. The prey population decays due to predation by the predator at a rate δ and the predation function is XY. Again, the predator population increases by consuming the preys at a rate β and decays due to natural death at a rate γ. The mathematical formulation of the model is

$$\begin{cases} \frac{dX(t)}{dt} = sX(t)(1 - \frac{X(t)}{K}) - \delta X(t)Y(t) \\ \frac{dY(t)}{dt} = \beta X(t)Y(t) - \gamma Y(t) \end{cases} \tag{6}$$

This can be expressed as

$$\begin{cases} \frac{dX(t)}{dt} = sX(t) - \frac{sX^2(t)}{K} - \alpha X(t)Y(t) \\ \frac{dY(t)}{dt} = \beta X(t)Y(t) - \gamma Y(t) \end{cases} \tag{7}$$

6 Formulation and stability analysis of the imprecise model

Along with the prey X and the predator Y population, it has been considered that the predation rate δ, growth rate s, death rate γ and increase rate β to be imprecise. So the reformulated system become

$$\begin{cases} \frac{d\widehat{X}(t)}{dt} = \widehat{s}\widehat{X}(t) - \frac{\widehat{s}\widehat{X}^2(t)}{K} - \widehat{\delta}\widehat{X}(t)\widehat{Y}(t) \\ \frac{d\widehat{Y}(t)}{dt} = \widehat{\beta}\widehat{X}(t)\widehat{Y}(t) - \widehat{\gamma}\widehat{Y}(t) \end{cases} \tag{8}$$

Now taking the imprecise variables and parameters to be interval numbers given by $\widehat{X} = [\underline{X}^\alpha, \overline{X}^\alpha]$, $\widehat{Y} = [\underline{Y}^\alpha, \overline{Y}^\alpha]$, $\widehat{\delta} = [\underline{\delta}^\alpha, \overline{\delta}^\alpha]$, $\widehat{\beta} = [\underline{\beta}^\alpha, \overline{\beta}^\alpha]$ and $\widehat{\gamma} = [\underline{\gamma}^\alpha, \overline{\gamma}^\alpha]$ and using Theorem-**??** we have $\dot{\widehat{X}} = [\underline{\dot{X}}^\alpha, \overline{\dot{X}}^\alpha]$ and $\dot{\widehat{Y}} = [\underline{\dot{Y}}^\alpha, \overline{\dot{Y}}^\alpha]$ for the first form or $\dot{\widehat{X}} = [\overline{\dot{X}}^\alpha, \underline{\dot{X}}^\alpha]$ and $\dot{\widehat{Y}} = [\overline{\dot{Y}}^\alpha, \underline{\dot{Y}}^\alpha]$ for the second form. Now by using "e" and "g" operator method, and following [7] the system reduced to the following sub section:

6.1 Case-I

$$\begin{cases} e(\underline{\dot{P}}^\alpha(t) + i\overline{\dot{P}}^\alpha(t)) = e(\underline{s}^\alpha + i\overline{s}^\alpha)e(\underline{P}^\alpha(t) + i\overline{P}^\alpha(t)) \\ \qquad -g(\underline{s}^\alpha + i\overline{s}^\alpha)e(\underline{P}^\alpha(t) + i\overline{P}^\alpha(t))e(\underline{P}^\alpha(t) + i\overline{P}^\alpha(t))/K \\ \qquad -g(\underline{\delta}^\alpha + i\overline{\delta}^\alpha)e(\underline{P}^\alpha(t) + i\overline{P}^\alpha(t))e(\underline{N}^\alpha(t) + i\overline{N}^\alpha(t)) \\ e(\underline{\dot{N}}^\alpha(t) + i\overline{\dot{N}}^\alpha(t)) = e(\underline{\beta}^\alpha + i\overline{\beta}^\alpha)e(\underline{P}^\alpha(t) + i\overline{P}^\alpha(t))e(\underline{N}^\alpha(t) + i\overline{N}^\alpha(t)) \\ \qquad -g(\underline{\gamma}^\alpha + i\overline{\gamma}^\alpha)e(\underline{N}^\alpha(t) + i\overline{N}^\alpha(t)) \end{cases} \tag{9}$$

where $\overline{X}^\alpha(t) = \max_{t\in[0,\infty)} \left\{\overline{P}^\alpha(t), \underline{P}^\alpha(t)\right\}$, $\underline{X}^\alpha(t) = \min_{t\in[0,\infty)} \left\{\overline{P}^\alpha(t), \underline{P}^\alpha(t)\right\}$, $\overline{Y}^\alpha(t) = \max_{t\in[0,\infty)} \left\{\overline{N}^\alpha(t), \underline{N}^\alpha(t)\right\}$, $\underline{Y}^\alpha(t) = \min_{t\in[0,\infty)} \left\{\overline{N}^\alpha(t), \underline{N}^\alpha(t)\right\}$.

6.2 Local Stability Analysis of Case-I

Form equation (9) it can be obtained that

$$\begin{cases} (\dot{\underline{P}}^{\alpha}(t) + i\dot{\overline{P}}^{\alpha}(t)) = (\underline{s}^{\alpha} + i\overline{s}^{\alpha})(\underline{P}^{\alpha}(t) + i\overline{P}^{\alpha}(t)) \\ \qquad -(\overline{s}^{\alpha} + i\underline{s}^{\alpha})(\underline{P}^{\alpha}(t) + i\overline{P}^{\alpha}(t))(\underline{P}^{\alpha}(t) + i\overline{P}^{\alpha}(t))/K \\ \qquad -(\overline{\delta}^{\alpha} + i\underline{\delta}^{\alpha})(\underline{P}^{\alpha}(t) + i\overline{P}^{\alpha}(t))(\underline{N}^{\alpha}(t) + i\overline{N}^{\alpha}(t)) \\ (\dot{\underline{N}}^{\alpha}(t) + i\dot{\overline{N}}^{\alpha}(t)) = (\underline{\beta}^{\alpha} + i\overline{\beta}^{\alpha})(\underline{P}^{\alpha}(t) + i\overline{P}^{\alpha}(t))(\underline{N}^{\alpha}(t) + i\overline{N}^{\alpha}(t)) \\ \qquad -(\overline{\gamma}^{\alpha} + i\underline{\gamma}^{\alpha})(\underline{N}^{\alpha}(t) + i\overline{N}^{\alpha}(t)) \end{cases} \tag{10}$$

Therefore, the following system of ODE can be obtained

$$\begin{cases} \dot{\underline{P}}^{\alpha}(t) = \underline{s}^{\alpha}\underline{P}^{\alpha}(t) - \overline{s}^{\alpha}\underline{P}^{\alpha 2}(t)/K - \overline{\delta}^{\alpha}\underline{P}^{\alpha}(t)\underline{N}^{\alpha}(t) \\ \dot{\overline{P}}^{\alpha}(t) = \overline{s}^{\alpha}\overline{P}^{\alpha}(t) - \underline{s}^{\alpha}\overline{P}^{\alpha 2}(t)/K - \underline{\delta}^{\alpha}\overline{P}^{\alpha}(t)\overline{N}^{\alpha}(t) \\ \dot{\underline{N}}^{\alpha}(t) = \underline{\beta}^{\alpha}\underline{P}^{\alpha}(t)\underline{N}^{\alpha}(t) - \overline{\gamma}^{\alpha}\underline{N}^{\alpha}(t) \\ \dot{\overline{N}}^{\alpha}(t) = \overline{\beta}^{\alpha}\overline{P}^{\alpha}(t)\overline{N}^{\alpha}(t) - \underline{\gamma}^{\alpha}\overline{N}^{\alpha}(t) \end{cases} \tag{11}$$

For simplicity let us consider the following:
$\underline{s}^{\alpha} = s_1^{\alpha},\ \overline{s}^{\alpha} = s_2^{\alpha},\ \underline{\delta}^{\alpha} = \delta_1^{\alpha},\ \overline{\delta}^{\alpha} = \delta_2^{\alpha},\ \underline{\beta}^{\alpha} = \beta_1^{\alpha},\ \overline{\beta}^{\alpha} = \beta_2^{\alpha},\ \underline{\gamma}^{\alpha} = \gamma_1^{\alpha},\ \overline{\gamma}^{\alpha} = \gamma_2^{\alpha}$
$\underline{P}^{\alpha} = P_1^{\alpha},\ \overline{P}^{\alpha} = P_2^{\alpha},\ \underline{N}^{\alpha} = N_1^{\alpha},\ \overline{N}^{\alpha} = N_2^{\alpha}$

The modified system of (11) is

$$\begin{cases} \dot{P_1}^{\alpha} = s_1^{\alpha}P_1^{\alpha} - \frac{s_2^{\alpha}P_1^{\alpha 2}}{K} - \delta_2^{\alpha}P_1{}^{\alpha}N_1^{\alpha} \\ \dot{P_2}^{\alpha} = s_2^{\alpha}P_2^{\alpha} - \frac{s_1^{\alpha}P_2^{\alpha 2}}{K} - \delta_1^{\alpha}P_2^{\alpha}N_2^{\alpha} \\ \dot{N_1}^{\alpha} = \beta_1^{\alpha}P_1^{\alpha}N_1^{\alpha} - \gamma_2^{\alpha}N_1^{\alpha} \\ \dot{N_2}^{\alpha} = \beta_2^{\alpha}P_2^{\alpha}N_2^{\alpha} - \gamma_1^{\alpha}N_2^{\alpha} \end{cases} \tag{12}$$

The steady state solutions are given by $P_1^{\alpha *} = \frac{\gamma_2{}^{\alpha}}{\beta_1{}^{\alpha}}, P_2^{\alpha *} = \frac{\gamma_1{}^{\alpha}}{\beta_2{}^{\alpha}}, N_1^{\alpha *} = \frac{s_1^{\alpha} - s_2^{\alpha}\gamma_2^{\alpha}/\beta_1^{\alpha}K}{\delta_2{}^{\alpha}}, N_2^{\alpha *} = \frac{s_2{}^{\alpha} - s_1{}^{\alpha}\gamma_1{}^{\alpha}/\beta_2{}^{\alpha}K}{\delta_1{}^{\alpha}}$.
The Jacobian matrix for the system in steady state $(P_1^{\alpha *}, P_2^{\alpha *}, N_1^{\alpha *}, N_2^{\alpha *})$ is given by

$$V = \begin{pmatrix} -\frac{\gamma_2{}^{\alpha}s_2{}^{\alpha}}{\beta_1{}^{\alpha}K} & 0 & -\frac{\delta_2{}^{\alpha}\gamma_2{}^{\alpha}}{\beta_1{}^{\alpha}} & 0 \\ 0 & -\frac{\gamma_1{}^{\alpha}s_1{}^{\alpha}}{\beta_2{}^{\alpha}K} & 0 & -\frac{\delta_1{}^{\alpha}\gamma_1{}^{\alpha}}{\beta_2{}^{\alpha}} \\ \frac{s_1{}^{\alpha}\beta_1{}^{\alpha}K - s_2{}^{\alpha}\gamma_2{}^{\alpha}}{K\delta_2{}^{\alpha}} & 0 & 0 & 0 \\ 0 & \frac{s_2{}^{\alpha}\beta_2{}^{\alpha}K - s_1{}^{\alpha}\gamma_1{}^{\alpha}}{K\delta_1{}^{\alpha}} & 0 & 0 \end{pmatrix}$$

and the corresponding eigenvalues are given by

$$-\frac{s_1{}^{\alpha}\gamma_1{}^{\alpha} \pm \sqrt{\gamma_1{}^{\alpha}(-4K^2 s_2{}^{\alpha}\beta_2{}^{\alpha} + s_1^{\alpha 2}\gamma_1^{\alpha} + 4Ks_1{}^{\alpha}\beta_2{}^{\alpha}\gamma_1{}^{\alpha})}}{2K\beta_2{}^{\alpha}},$$
$$-\frac{s_2{}^{\alpha}\gamma_2{}^{\alpha} \pm \sqrt{\gamma_2{}^{\alpha}(-4K^2 s_1{}^{\alpha}\beta_1{}^{\alpha} + s_2^{\alpha 2}\gamma_2^{\alpha} + 4Ks_2{}^{\alpha}\beta_1{}^{\alpha}\gamma_2{}^{\alpha})}}{2K\beta_1{}^{\alpha}}.$$

Clearly, the all the eigenvalues are negative if

$$\sqrt{\gamma_1{}^\alpha(-4K^2 s_2{}^\alpha \beta_2^{2\alpha} + s_1^{\alpha 2}\gamma_1{}^\alpha + 4Ks_1{}^\alpha\beta_2{}^\alpha\gamma_1{}^\alpha)} < s_1{}^\alpha\gamma_1{}^\alpha \text{and}, \tag{13}$$

$$\sqrt{\gamma_2{}^\alpha(-4K^2 s_1{}^\alpha \beta_1^{\alpha 2} + s_2^{\alpha 2}\gamma_2{}^\alpha + 4Ks_2{}^\alpha\beta_1{}^\alpha\gamma_2{}^\alpha)} < s_2{}^\alpha\gamma_2{}^\alpha. \tag{14}$$

The above theory is illustrated in the following numerical experiment:

6.3 Numerical Experiment Case-I

The parameters are assumed as which shows the steady state solutions are stable.

Table 1. Input data

$s_1{}^\alpha$	1.1	$\delta_1{}^\alpha$	0.01	$\beta_1{}^\alpha$	.001	$\gamma_1{}^\alpha$	0.085	K	150
$s_2{}^\alpha$	1.15	$\delta_2{}^\alpha$	0.015	$\beta_2{}^\alpha$	0.0015	$\gamma_2{}^\alpha$	0.090		

Table 2. Output Data

$P_1^{\alpha *}$	90	$N_1^{\alpha *}$	27.33
$P_2^{\alpha *}$	56.67	$N_2^{\alpha *}$	73.44

Table 3. Eigenvalues

-0.631575	$-0.207778 + 0.132856i$	$-0.207778 - 0.132856i$	$-0.0584254.$

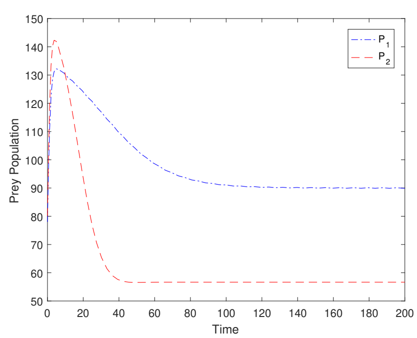

Fig. 1. Time series plot of prey population showing $\overline{X}^\alpha(t) = \max_{t\in[0,\infty)} \{P_1(t), P_2(t)\}$, $\underline{X}^\alpha(t) = \min_{t\in[0,\infty)} \{P_1(t), P_2(t)\}$

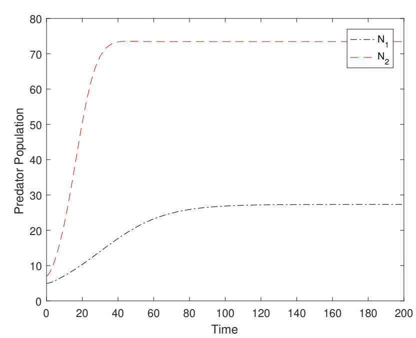

Fig. 2. Time series plot of predator population showing $\overline{Y}^\alpha(t) = \max_{t\in[0,\infty)} \{N_1(t), N_2(t)\}$, $\underline{Y}^\alpha(t) = \min_{t\in[0,\infty)} \{N_1(t), N_2(t)\}$

6.4 Case-II

$$\begin{cases} g(\dot{\underline{P}}^{\alpha}(t)+i\dot{\overline{P}}^{\alpha}(t)) = g(\underline{s}^{\alpha}+i\overline{s}^{\alpha})g(\underline{P}^{\alpha}(t)+i\overline{P}^{\alpha}(t)) \\ \qquad -g(\underline{s}^{\alpha}+i\overline{s}^{\alpha})g(\underline{P}^{\alpha}(t)+i\overline{P}^{\alpha}(t))g(\underline{P}^{\alpha}(t)+i\overline{P}^{\alpha}(t))/K \\ \qquad -g(\underline{\delta}^{\alpha}+i\overline{\delta}^{\alpha})g(\underline{P}^{\alpha}(t)+i\overline{P}^{\alpha}(t))g(\underline{N}^{\alpha}(t)+i\overline{N}^{\alpha}(t)) \\ g(\dot{\underline{N}}^{\alpha}(t)+i\dot{\overline{N}}^{\alpha}(t)) = e(\underline{\beta}^{\alpha}+i\overline{\beta}^{\alpha})g(\underline{P}^{\alpha}(t)+i\overline{P}^{\alpha}(t))g(\underline{N}^{\alpha}(t)+i\overline{N}^{\alpha}(t)) \\ \qquad -e(\underline{\gamma}^{\alpha}+i\overline{\gamma}^{\alpha})g(\underline{N}^{\alpha}(t)+i\overline{N}^{\alpha}(t)) \end{cases} \tag{15}$$

where $\overline{X}^{\alpha}(t)=\max\limits_{t\in[0,\infty)}\left\{\overline{P}^{\alpha}(t),\underline{P}^{\alpha}(t)\right\}$, $\underline{X}^{\alpha}(t)=\min\limits_{t\in[0,\infty)}\left\{\overline{P}^{\alpha}(t),\underline{P}^{\alpha}(t)\right\}$, $\overline{Y}^{\alpha}(t)=\max\limits_{t\in[0,\infty)}\left\{\overline{N}^{\alpha}(t),\underline{N}^{\alpha}(t)\right\}$, $\underline{Y}^{\alpha}(t)=\min\limits_{t\in[0,\infty)}\left\{\overline{N}^{\alpha}(t),\underline{N}^{\alpha}(t)\right\}$.

The stability of the above two systems given by (9) and (15) can be analysed as following:

6.5 Local Stability Analysis of Case-II

Form equation (15) it can be obtained that

$$\begin{cases} (\dot{\overline{P}}^{\alpha}(t)+i\dot{\underline{P}}^{\alpha}(t)) = (\overline{s}^{\alpha}+i\underline{s}^{\alpha})(\overline{P}^{\alpha}(t)+i\underline{P}^{\alpha}(t)) \\ \qquad -(\overline{s}^{\alpha}+i\underline{s}^{\alpha})(\overline{P}^{\alpha}(t)+i\underline{P}^{\alpha}(t))(\overline{P}^{\alpha}(t)+i\underline{P}^{\alpha}(t))/K \\ \qquad -(\overline{\delta}^{\alpha}+i\underline{\delta}^{\alpha})(\overline{P}^{\alpha}(t)+i\underline{P}^{\alpha}(t))(\overline{N}^{\alpha}(t)+i\underline{N}^{\alpha}(t)) \\ (\dot{\overline{N}}^{\alpha}(t)+i\dot{\underline{N}}^{\alpha}(t)) = (\underline{\beta}^{\alpha}+i\overline{\beta}^{\alpha})(\overline{P}^{\alpha}(t)+i\underline{P}^{\alpha}(t))(\overline{N}^{\alpha}(t)+i\underline{N}^{\alpha}(t)) \\ \qquad -(\underline{\gamma}^{\alpha}+i\overline{\gamma}^{\alpha})(\overline{N}^{\alpha}(t)+i\underline{N}^{\alpha}(t)) \end{cases} \tag{16}$$

Therefore we have the following system of ODE

$$\begin{cases} \dot{\overline{P}}^{\alpha}(t)=\overline{s}^{\alpha}\overline{P}^{\alpha}(t)-\overline{s}^{\alpha}\overline{P}^{\alpha 2}(t)/K-\overline{\delta}^{\alpha}\overline{P}^{\alpha}(t)\overline{N}^{\alpha}(t) \\ \dot{\underline{P}}^{\alpha}(t)=\underline{s}^{\alpha}\underline{P}^{\alpha}(t)-\underline{s}^{\alpha}\underline{P}^{\alpha 2}(t)/K-\underline{\delta}^{\alpha}\underline{P}^{\alpha}(t)\underline{N}^{\alpha}(t) \\ \dot{\overline{N}}^{\alpha}(t)=\underline{\beta}^{\alpha}\overline{P}^{\alpha}(t)\overline{N}^{\alpha}(t)-\underline{\gamma}^{\alpha}\overline{N}^{\alpha}(t) \\ \dot{\underline{N}}^{\alpha}(t)=\overline{\underline{\beta}}^{\alpha}\underline{P}^{\alpha}(t)\underline{N}(t)^{\alpha}-\overline{\gamma}^{\alpha}\underline{N}^{\alpha}(t) \end{cases} \tag{17}$$

For simplicity let us consider the following:
$\underline{s}^{\alpha}=s_1^{\alpha},\ \overline{s}^{\alpha}=s_2^{\alpha},\ \underline{\delta}^{\alpha}=\delta_1^{\alpha},\ \overline{\delta}^{\alpha}=\delta_2^{\alpha},\ \underline{\beta}^{\alpha}=\beta_1^{\alpha},\ \overline{\beta}^{\alpha}=\beta_2^{\alpha},\ \underline{\gamma}^{\alpha}=\gamma_1^{\alpha},\ \overline{\gamma}^{\alpha}=\gamma_2^{\alpha}$
$\underline{P}^{\alpha}=P_1^{\alpha},\ \overline{P}^{\alpha}=P_2^{\alpha},\ \underline{N}^{\alpha}=N_1^{\alpha},\ \overline{N}^{\alpha}=N_2^{\alpha}$ The modified system of (17) is

$$\begin{cases} \dot{P_1}^{\alpha}=s_1^{\alpha}P_1^{\alpha}-\frac{s_1{}^{\alpha}P_1^{\alpha 2}}{K}-\delta_1{}^{\alpha}P_1{}^{\alpha}N_1{}^{\alpha} \\ \dot{P_2}^{\alpha}=s_2^{\alpha}P_2^{\alpha}-\frac{s_2{}^{\alpha}P_2^{\alpha 2}}{K}-\delta_2{}^{\alpha}P_2{}^{\alpha}N_2{}^{\alpha} \\ \dot{N_1}^{\alpha}=\beta_2^{\alpha}P_1^{\alpha}N_1^{\alpha}-\gamma_2^{\alpha}N_1^{\alpha} \\ \dot{N_2}^{\alpha}=\beta_1^{\alpha}P_2^{\alpha}N_2^{\alpha}-\gamma_1^{\alpha}N_2^{\alpha} \end{cases} \tag{18}$$

The steady state solutions are given by $P_1^{\alpha *}=\frac{\gamma_2{}^{\alpha}}{\beta_2{}^{\alpha}}, P_2^{\alpha *}=\frac{\gamma_1{}^{\alpha}}{\beta_1{}^{\alpha}}, N_1^{\alpha *}=\frac{s_1{}^{\alpha}-s_1{}^{\alpha}\gamma_2{}^{\alpha}/\beta_2{}^{\alpha}K}{\delta_1{}^{\alpha}}, N_2^{\alpha *}=\frac{s_2{}^{\alpha}-s_2{}^{\alpha}\gamma_1{}^{\alpha}/\beta_1{}^{\alpha}K}{\delta_2{}^{\alpha}}$.

The Jacobian matrix for the system in steady state $(P_1^{\alpha *}, P_2^{\alpha *}, N_1^{\alpha *}, N_2^{\alpha *})$ is given by

$$V = \begin{pmatrix} -\frac{\gamma_2{}^{\alpha} s_1{}^{\alpha}}{\beta_2{}^{\alpha} K} & 0 & -\frac{\delta_2{}^{\alpha}\gamma_1{}^{\alpha}}{\beta_1{}^{\alpha}} & 0 \\ 0 & -\frac{\gamma_1{}^{\alpha} s_2{}^{\alpha}}{\beta_1{}^{\alpha} K} & 0 & -\frac{\delta_2{}^{\alpha}\gamma_1{}^{\alpha}}{\beta_1{}^{\alpha}} \\ \frac{s_1{}^{\alpha}\beta_2{}^{\alpha}K - s_1{}^{\alpha}\gamma_2{}^{\alpha}}{K\delta_1{}^{\alpha}} & 0 & 0 & 0 \\ 0 & \frac{s_2{}^{\alpha}\beta_1{}^{\alpha}K - s_2{}^{\alpha}\gamma_1{}^{\alpha}}{K\delta_2{}^{\alpha}} & 0 & 0 \end{pmatrix}$$

and the corresponding eigen values are given by

$$-\frac{s_2{}^{\alpha}\gamma_1{}^{\alpha} \pm \sqrt{s_2{}^{\alpha}\gamma_1{}^{\alpha}(-4K^2\beta_1^{\alpha 2} + s_2{}^{\alpha}\gamma_1{}^{\alpha} + 4K\beta_1{}^{\alpha}\gamma_1{}^{\alpha})}}{2K\beta_1{}^{\alpha}},$$
$$-\frac{s_1{}^{\alpha}\gamma_2{}^{\alpha} \pm \sqrt{s_1{}^{\alpha}\gamma_2{}^{\alpha}(-4K^2\beta_2^{2\alpha} + s_1{}^{\alpha}\gamma_2{}^{\alpha} + 4K\beta_2{}^{\alpha}\gamma_2{}^{\alpha})}}{2K\beta_2{}^{\alpha}}.$$

Clearly, in this case also all the eigen values are negative if

$$\sqrt{s_2{}^{\alpha}\gamma_1{}^{\alpha}(-4K^2\beta_1^{\alpha 2} + s_2{}^{\alpha}\gamma_1{}^{\alpha} + 4K\beta_1{}^{\alpha}\gamma_1{}^{\alpha})} < s_2{}^{\alpha}\gamma_1{}^{\alpha}, \tag{19}$$
$$\sqrt{s_1{}^{\alpha}\gamma_2{}^{\alpha}(-4K^2\beta_2^{\alpha 2} + s_1{}^{\alpha}\gamma_2{}^{\alpha} + 4K\beta_2{}^{\alpha}\gamma_2{}^{\alpha})} < s_1{}^{\alpha}\gamma_2{}^{\alpha}. \tag{20}$$

Let us have the following numerical experiment to illustrate the above.

6.6 Numerical Experiment-2

The input parameters are assumed to be the same as in numerical experiment-1 in 6.3.

Table 4. Output Data

$P_1^{\alpha *}$	60	$N_1^{\alpha *}$	66
$P_2^{\alpha *}$	85	$N_2^{\alpha *}$	23.22

Table 5. Eigenvalues

-0.578438	$-0.22 + 0.104881i$	$-0.22 - 0.104881i$	-0.0732288

which shows the steady state solutions are stable. The graphs for the above case are given by the following.

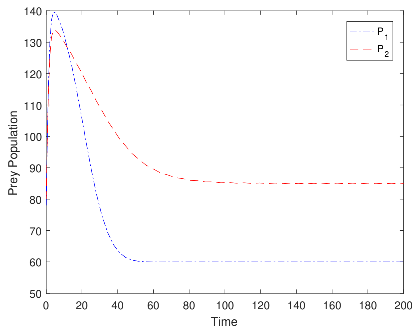

Fig. 3. Time series plot of prey population showing $\overline{X}^{\alpha}(t) = \max\limits_{t\in[0,\infty)} \{P_1(t), P_2(t)\}$, $\underline{X}^{\alpha}(t) = \min\limits_{t\in[0,\infty)} \{P_1(t), P_2(t)\}$

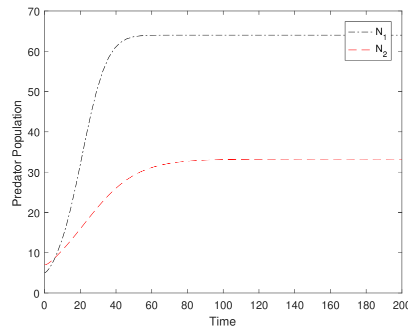

Fig. 4. Time series plot of predator population showing $\overline{Y}^{\alpha}(t) = \max\limits_{t\in[0,\infty)} \{N_1(t), N_2(t)\}$, $\underline{Y}^{\alpha}(t) = \min\limits_{t\in[0,\infty)} \{N_1(t), N_2(t)\}$

7 Discussion

In this work, we have analyzed a prey–predator model in imprecise environment. We have considered both prey and predator population as imprecise variables. Also, we have assumed the parameters namely growth rate of prey, predation rate of the prey population, increase rate and decay rate of the predator population as imprecise parameters. The imprecise model converted to two equivalent crisp models given by Equations (9) and (15). The non-zero steady state solutions are obtained for each of the cases. The local stability analysis is done with the help of eigen values of the jacobian matrices.

It can be observed form the system given by Equation(9) is stable if the system satisfies the condition given by Equation(13).
The numerical experiment-1 in 6.3 corresponds to the system Equation (9). Table-1 shows the input data, Table-2 presents the output data and Table-3 gives the eigen values of the jacobian matrix at the steady state. From table-3 we observe that the eigen values have negative real part, so the steady state solutions are stable.

Figures 1 and 2 represents time series plot for prey and predator populations respectively. It can be observed from both the figures that the graphical solutions are at a good agrement with the numerical values.

Similarly, form the system given by Equation(15) is stable if the system satisfies the condition given by (19).
The numerical experiment-2 in 6.6 corresponds to the system (15). With the same the input data as on Numerical experiment-1, Table-4 presents the output data and Table-5 gives the eigen values of the jacobian matrix at the steady state. From table-5 we observe that the eigen values have negative real part, so the steady state solutions are stable.

Figures 3 and 4 represents time series plot for prey and predator populations respectively. It can be observed from both the figures that the graphical solutions are at a good agrement with the numerical values.

In figure 1, we observe that the solution curves corresponding to the lower and upper values of the prey population for system (9), do not always remain as lower and upper values, rather the curves corresponding to lower value becomes upper value and the upper value becomes the lower value.

Again, from figure 4, it has been analyzed that the solution curves corresponding to the upper and lower values of the predator populations for system (15), do not always remain as upper and lower values, but the curves corresponding to lower value becomes upper value and the upper value becomes the lower value. Furthermore, it can be observed that in place of a single curve for each population, we are obtaining a lower and upper boundaries of the stable solutions. In case of interval approach to manipulate this type of problem we need to check the solution curves for different parametric values, but in this case we have obtained the boundaries at a single attempt. Though, we have checked the stability of only two cases, but we can obtain some more different cases with different combinations of “e” and “g” operators.

8 Conclusion

A prey-predator model is developed in the model in imprecise environment. Due to the environmental variation in different ecological conditions, the natural parameters vary. Some of the researchers developed imprecise models considering the natural parameters to be imprecise. These imprecise parameter are converted to interval numbers and depending upon the different parametric conditions the problems are solved. In the present work, the parameters like growth rate of prey, predation rate of the prey population, increase rate and decay rate of the predator population along with the prey and predator population are assumed to be imprecise. The imprecise model then converted to equivalent two different crisp model with the help of “e” and “g” operator method. Stability for both the crisp model are analyzed theoretically. With the help of numerical examples both the models are presented numerically and graphically. Numerical results and the graphical analysis for both the crisp model provides the upper and lower boundaries of the stable solutions for both the population rather than a single solution curve.

For different combinations of “e” and “g” we can obtain some more cases in crisp form. The model can be extended to a prey-predator harvesting model, prey-predator harvesting model with budget constraints etc..

References

1. Chakraborty, Kunal, Das, Sanjoy and Kar T. K. Optimal control of effort of a stage structured prey–predator fishery model with harvesting. *Nonlinear Analysis: Real World Applications*, 12(6):3452–3467, 2011.
2. da, Silva Peixoto, Magda, cio Carvalho de Barros,Laé, and Carlos, Rodney Bassanezi Predator–prey fuzzy model. *Ecological Modelling*, 214(1):39–44, 2008.
3. De,A, Maity, K, and Maiti, M. Stability analysis of combined project of fish, broiler and ducks: Dynamical system in imprecise environment. *International Journal of Biomathematics*, 8(05):1550067, 2015.
4. De,A, Maity, K, and Maiti, M. Fish and broiler optimal harvesting models in imprecise environment. *International Journal of Biomathematics*, 10(08):1750115, 2017.
5. Guo, Mengshu, Xue, Xiaoping, and Li Ronglu. Impulsive functional differential inclusions and fuzzy population models. *Fuzzy Sets and Systems*, 138(3):601–615, 2003.
6. Kar, Tapan Kumar Modelling and analysis of a harvested prey–predator system incorporating a prey refuge. *Journal of Computational and Applied Mathematics*, 185(1):19–33, 2006.
7. Khatua, Debnarayan, and Maity, Kalipada Stability of fuzzy dynamical systems based on quasi-level-wise system. *Journal of Intelligent & Fuzzy Systems*, 33(6):3515–3528, 2017.
8. Lotka, Alfred James *Elements of Physical Biology.* Williams and Wilkins Company, 1925.
9. Ma, Ming, Friedman, Menahem, and Kandel, Abraham A new fuzzy arithmetic. *Fuzzy Sets and Systems*, 108(1):83–90, 1999.
10. Pal, D, Mahapatra, G. S., and Samanta, G, P. Stability and bionomic analysis of fuzzy parameter based prey–predator harvesting model using ufm. *Nonlinear Dynamics*, 79(3):1939–1955, 2015.
11. Pal, D, Mahapatra, G. S., and Samanta, G, P. Optimal harvesting of prey–predator system with interval biological parameters: a bioeconomic model. *Mathematical Biosciences*, 241(2):181–187, 2013.
12. Puri, Madan L and Ralescu, Dan A Differentials of fuzzy functions. *Journal of Mathematical Analysis and Applications*, 91(2):552–558, 1983.
13. Qu, Ying and Wei, Junjie Bifurcation analysis in a time-delay model for prey–predator growth with stage-structure. *Nonlinear Dynamics*, 49(1-2):285–294, 2007.
14. Seo, Gunog and DeAngelis, Donald L A predator–prey model with a holling type i functional response including a predator mutual interference. *Journal of Nonlinear Science*, 21(6):811–833, 2011.
15. Tapaswini, Smita and Chakraverty, S Numerical solution of fuzzy arbitrary order predator-prey equations. *Applications & Applied Mathematics*, 8(2), 2013.
16. Volterra, P Lessons in the mathematical theory of the struggle for life. *Paris: Seuil*, 1931.
17. Wang Yuangan Stability analysis of predator-prey system with fuzzy impulsive control. *Journal of Applied Mathematics*, 2012, 2012.