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Abstract. In this paper, the traffic flow problem is converted into a system stability problem through variable substitution and a phase plane analysis method is presented for analyzing the complex nonlinear traffic phenomena. This method matches traffic congestion with the unstable system. So these theories and methods of stability can be applied directly to solve the traffic problem. Based on an anisotropic continuum model developed by Gupta and Katiyar(GK model), this paper uses this new method to describe various nonlinear phenomena due to different input and output conditions on ramps which were rarely studied in the past. The results show that the traffic phenomena described by the new method is consistent with that described by traditional methods. Moreover, the phase plane diagram highlights the unstable traffic phenomena we are chiefly concerned about and describes the variation of density or velocity with time or sections more clearly.

Keywords: Phase Plane Diagrams, Nonlinear Traffic Phenomena, Stability Analysis, Ramps.

1 Introduction

In a real traffic flow, almost every driver meets with the phenomenon of traffic congestion. There have been several recent advances in traffic theory, notably those that treat traffic like a fluid. An important branch of the subject, with repercussions on all the other branches, is the quantitative study of traffic phenomena. More recently, there has been an increasing tendency to adopt scientific methods, and try to assess all kinds of traffic phenomena by means of controlled experiments. Paralleled with ex-

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periments, many physical models have been proposed [1-3]. Most of them are hydrodynamic models which provide a macroscopic description of traffic flow. The study of continuum traffic phenomena began with the LWR model developed independently by Lighthill and Whitham [4] and Richards [5]. In this model, vehicles have often been considered as interacting particles and traffic flow can be considered as a one-dimensional compressible flow of these particles. In the past decades, researchers have made many efforts to improve the LWR model, they developed many higher order models which use a dynamic equation to make speed to replace the equilibrium relationship. Subsequent studies [6-9] of the models have explained many observed features of the free flow and traffic jams in highways. Gupta and Katiyar[10]develop a macroscopic continuum traffic flow model to solve the characteristic speed problem that exists in the previously developed high-order models, which is referred to as GK model

In this paper, we use a new method to describe a variety of nonlinear traffic flow phenomena which are raised by different input and output on the ramp. We use some variable substitution to convert the traffic flow model into a functional stability model. From this model many well-known nonlinear phenomena may be analyzed. This paper studies the change of the flow at the ramp on the highway which is rarely studied by others. It includes various situations of fixed vehicle generation rate but increasing initial homogeneous density with a single ramp.

The remainder of the paper is organized as follows. In Section 2, we present the description of variable substitution and a functional stability model about traffic flow has been postulated. In Section 3, we analyze all kinds of nonlinear phenomena which are raised by different input and output on ramp by the new model. In Section 4, we concluded the paper.

2 Variable substitution based on GK model

GK model is an anisotropic continuous traffic flow model. It has been mostly studied nowadays and has the following form:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial(\rho v)}{\partial x} = 0 \\ \dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = a\left[V_e(\rho) - v\right] + aV_e'(\rho)\left[\dfrac{1}{2\rho}\dfrac{\partial \rho}{\partial x} + \dfrac{1}{6\rho^2}\dfrac{\partial^2 \rho}{\partial x^2} - \dfrac{1}{2\rho^3}\left(\dfrac{\partial \rho}{\partial x}\right)^2\right] \text{-} 2\beta c(\rho)\dfrac{\partial v}{\partial x} \end{cases} \tag{1}$$

where ρ is the density; υ is the velocity; x and t represent space and time respectively; a is the driver's sensitivity which equals the inverse of the driver's reaction time; $V_e\left[\rho(x,t)\right]$ is the optimal velocity function and has the following form:

$$V_e[\rho] = \mathrm{v}_f\left\{\left[1+\exp\left(\frac{\rho/\rho_m - 0.25}{0.06}\right)\right]^{-1} - 3.72\times 10^{-6}\right\} \tag{2}$$

$V_e'(\rho)=\dfrac{dV_e(\rho)}{d\rho}$, β is a non-negative dimensionless parameter and $c(\rho)\prec 0$ is the traffic sound speed given by:

$$c^2(\rho)=-\frac{aV_e^{'}(\rho)}{2},\ c=-\sqrt{-\frac{V_e^{'}(\rho)}{2\tau}}\ . \tag{3}$$

Here v_f is the free-flow speed, ρ_m is the maximum or jam density.

In the present paper, a simple transformation is employed as follow:

$$\begin{cases}\sigma=\dfrac{1}{v}\\ \eta=\dfrac{1}{\rho_m\text{-}\rho}\end{cases} \tag{4}$$

Substituting the variables into equation (1), we have a new traffic flow model as follow:

$$\begin{cases}\sigma^2\dfrac{\partial\eta}{\partial t}+\left(\eta-\rho_m\eta^2\right)\dfrac{\partial\sigma}{\partial x}+\sigma\dfrac{\partial\eta}{\partial x}=0\\ \dfrac{\partial\sigma}{\partial t}+\left(\dfrac{1}{\sigma}-2\beta\sqrt{-\dfrac{v_e'(\eta)}{2\tau}}\right)\dfrac{\partial\sigma}{\partial x}+a\sigma^2v_e'(\eta)\left\{\begin{array}{l}\dfrac{1}{2\eta\left(\rho_m\eta-1\right)}\cdot\dfrac{\partial\eta}{\partial x}+\dfrac{1}{6\left(\rho_m\eta-1\right)^2}\cdot\dfrac{\partial^2\eta}{\partial x^2}\\ -\left(\dfrac{1}{3\eta\left(\rho_m\eta-1\right)^2}+\dfrac{1}{2\eta\left(\rho_m\eta-1\right)^3}\right)\left(\dfrac{\partial\eta}{\partial x}\right)^2\end{array}\right\}+a\sigma^2v_e(\eta)-a\sigma=0\end{cases} \tag{5}$$

Similarly, substituting the variables into equation (2), the equilibrium velocity $v_e(\eta)$ is as follow:

$$V_e(\eta)=v_f\left\{\left[1+\exp\left(\frac{0.75\text{-}\dfrac{1}{\eta\rho_m}}{0.06}\right)\right]^{-1}-3.72\times10^{-6}\right\} \tag{6}$$

According to the variable substitution $\sigma=\dfrac{1}{v}$ and $\eta=\dfrac{1}{\rho_m\text{-}\rho}$, we can see that as long as the vehicles velocity goes to zero or the vehicle density becomes saturated, the state variable σ or η will approach infinity. So we can use the phase plane diagrams about the variable η or σ to describe clearly the relationship between traffic jams and system instability. When the traffic becomes congested, the state variable ρ and υ both tend to a specific value. However, through such variable substitutions, the state variable η or σ both tends to infinity. As long as there is traffic jam formation, the value of η or σ will approach infinity. So the problem of traffic flow could be converted into that of system stability. Some stability theories and mathematical tools can be applied directly to solve the traffic problems. If we use the new model by such variable substitution, we can see from the phase plane that there is a one-to-one relationship between the traffic congestion and the unstable system. The new traffic flow model can analyze various traffic phenomena directly and can also analyze the chaotic fluctuations of traffic flow.

3 The analysis of different input and output traffic phenomena using the new method

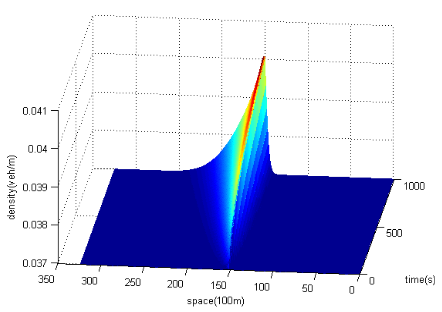
Gupta By analyzing the model, we first carried out numerical tests for the phenomena of fixed vehicle generation rate but increasing initial homogeneous density with a single ramp, which is rarely studied in the past. To study the effects of ramps, we added the source and the drain terms on the right-hand side of the continuity equation in (5) as follow:

$$\sigma^2 \frac{\partial \eta}{\partial t} + \left(\eta - \rho_m \eta^2\right) \frac{\partial \sigma}{\partial x} + \sigma \frac{\partial \eta}{\partial x} = r_{in}(t) - r_{out}(t) \tag{7}$$

where $r_{in}(t)$ and $r_{out}(t)$ represent the external flux through an on-ramp and through an off-ramp, respectively. This section uses MATLAB software to carry out numerical simulation in the Windows system environment. We have also taken the test road section as 32.2 km long and set a ramp in the middle of the road section. The vehicle generation rate was set to 0.0001veh/m/s. That was to say, the number of vehicles through an on-ramp was 0.0001 more than that through an off-ramp every meter per second. The initial density ρ_0 was 0.037veh/m. Other parameter values used were as follows:

$$\beta = 2.0, \tau = 14\text{s}, \text{v}_f = 30\text{m/s}, \rho_m = 0.2\text{veh/m} \tag{8}$$

The results were shown in Fig.1 (a)–(e).



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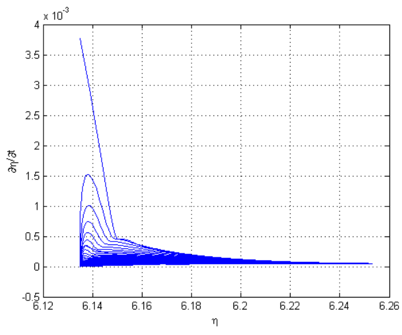

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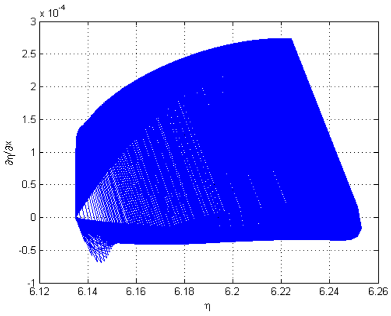

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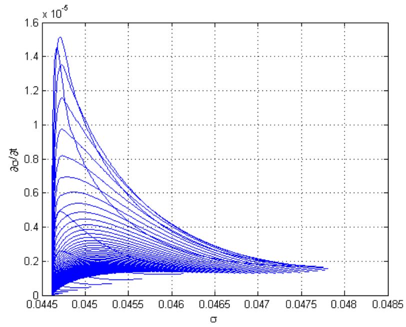

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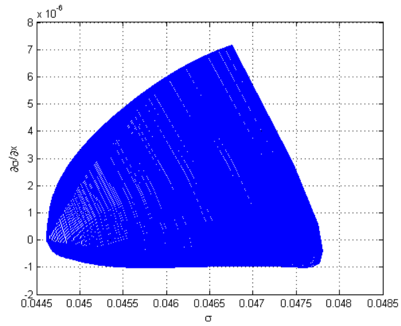


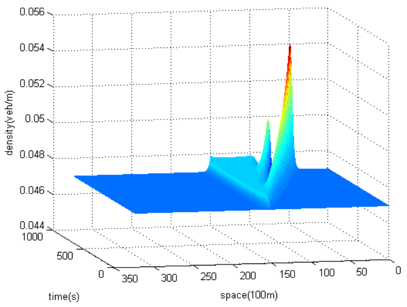
(e)

Fig. 1. The density temporal evolution and phase plane diagrams with initial homogeneous traffic of amplitude ρ_0 =0.037veh/m (a) the temporal evolution of vehicle density; (b) the phase plane diagram of $\left(\eta, \partial\eta/\partial t\right)$; (c) the phase plane diagram of $\left(\eta, \partial\eta/\partial x\right)$; (d) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial t\right)$; (e) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial x\right)$

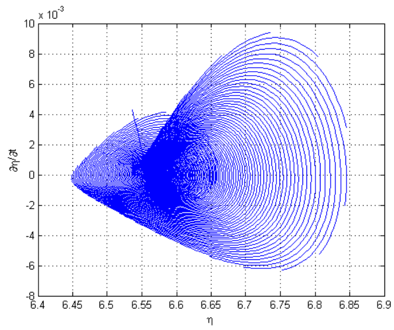
In Fig.1 (a), the vehicles come from the on-ramp will have an effect on the upstream traffic. Vehicles in upstream of the road need to decelerate when they move to the ramp and can't drive keeping the original speed. So the density in upstream of the road near the ramp will increase gradually. On the other hand, the initial density in downstream of the road also increases and the vehicles come from the on-ramp can't move downstream quickly, so some of them stay on the ramp, which makes the density increment on the ramp is a lot larger than that with a small initial density. The increment will decrease very fast as vehicles in downstream road sections move forward.

Fig.1 (b) is the combination of variation curves of η on each road section during the first 16 minutes. All curves in the figure change from left to right. That is to say, the value of η which is proportional to the vehicle density keeps on increasing. So, it mainly reflects the phenomenon that the density of each road section near the ramp increased gradually with time when vehicles continually entered from the ramp. Fig.1(c) could be considered as a group of curves which describe the change of η per second on the whole road. All curves are closed curves which increase firstly then decrease again. It illustrates that the upstream density near the ramp increases gradually and the downstream density decreases as vehicles move forward. Similarly, Fig.1 (d)-(e) reflect the same phenomena based on the velocity variation with time and displacement. Moreover, both the value and the change rate of η and σ in Fig.1 (b)-(e) increase more significantly than in the Fig.1 whose initial density is small.

If we continually increased the initial density to 0.047veh / m and remained other conditions such as the value of vehicle generation rate unchanged, the temporal evolution of vehicle density and phase plane diagrams could be compared as follows:



(a)



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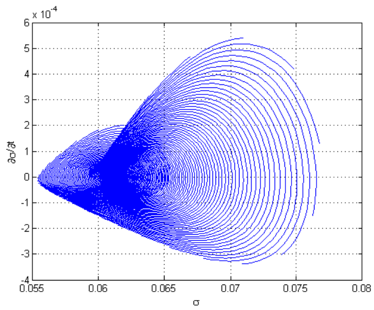

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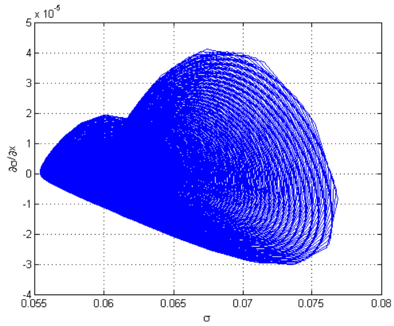

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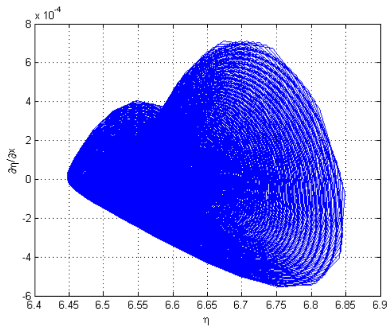
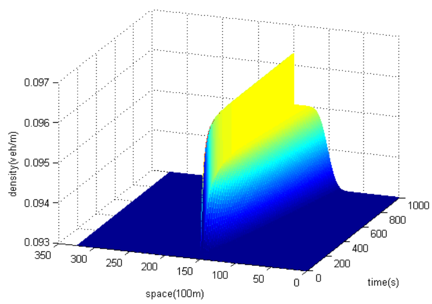


(e)

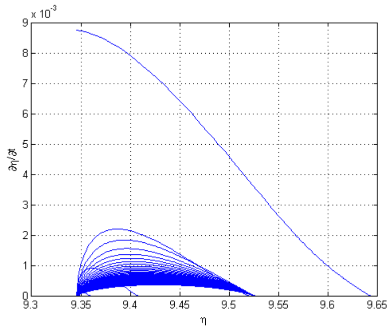
Fig. 2. The density temporal evolution and phase plane diagrams with initial homogeneous traffic of amplitude ρ_0 =0.047veh/m (a) the temporal evolution of vehicle density; (b) the phase plane diagram of $\left(\eta, \partial\eta/\partial t\right)$; (c) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial t\right)$; (d) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial x\right)$; (e) the phase plane diagram of $\left(\eta, \partial\eta/\partial x\right)$

Since the initial density is just above the down-critical unstable density, a small quantity of vehicles come from the ramp can be seen as a small localized perturbation on the initial homogeneous traffic flow and the amplitude of it grows in time, which eventually forms the stop-and-go traffic. The fluctuation amplitude of traffic flow is so large that the small vehicle generation rate on ramp can't make an appreciable effect on the density of the whole road. Fig.2 (b)-(e) all consist of number of circles and clearly highlight the fluctuations of density and velocity with time or displacement. So they also describe the stop-and-go traffic phenomenon.

If we continually increased the initial density to 0.093veh / m and remained other conditions unchanged, the temporal evolution of vehicle density and phase plane diagrams were shown as follows:



(a)



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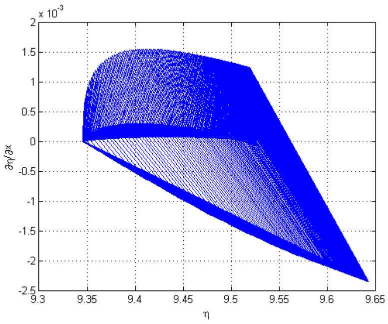

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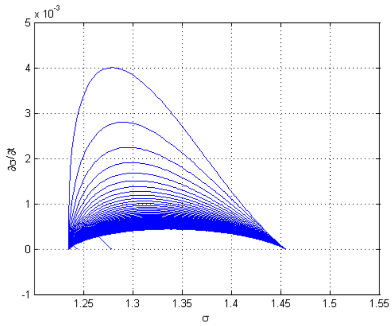

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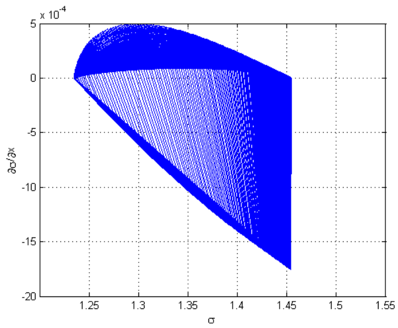

(e)

Fig. 3. The density temporal evolution and phase plane diagrams with initial homogeneous traffic of amplitude ρ_0 =0.093veh/m (a) the temporal evolution of vehicle density; (b) the phase plane diagram of $\left(\eta, \partial\eta/\partial t\right)$; (c) the phase plane diagram of $\left(\eta, \partial\eta/\partial x\right)$; (d) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial t\right)$; (e) the phase plane diagram of $\left(\sigma, \partial\sigma/\partial x\right)$

Fig.3 (a) shows that when the initial density becomes greater than the up-critical density, a stable regime of the model is reached again. The perturbation is dissipated and the vehicles come from the ramp will again make an appreciable effect on the whole road.

As the initial density increases even further, the vehicles come from the on-ramp will have more impact on the upstream traffic. Vehicles in upstream of the road also need to decelerate when they move to the ramp. So the density in upstream of the road near the ramp will increase largely.

On the other hand, the initial density in downstream of the road also increases and the vehicles come from the on-ramp can't move downstream quickly, so many of them stay on the ramp, which makes the density increment on the ramp increases larger. The increment will decrease very fast as vehicles in downstream road section move forward. Moreover, the value of vehicle generation rate is small but the initial density in downstream of the road is large. So the density increment in downstream road section is not obvious when the vehicles come from the ramp move downstream. It also can be seen from the Fig.3 (b)-(e) that no curves toward infinity are found in them. The whole road section is not blocked and the system is stable.

Fig.3 (b) is the combination of variation curves of η on each road section during the first 16 minutes. Similarly, all curves in the figure change from left to right. That is to say, the value of η which is proportional to the vehicle density keeps on increasing. So, it mainly reflects the phenomenon that the density of each road section increase gradually with time when vehicles continually enter the ramp. Fig.3 (c) could be considered as a group of curves which describe the change of η per second on the whole road. All curves are closed curves which increase firstly then decrease again. It illustrates that the upstream density near the ramp increases gradually and the downstream density decreases as vehicles move forward. Similarly, Fig.3 (d) and Fig.3 (e) reflect the same phenomena based on the velocity variation with time and displacement. Although the value and the change rate of η and σ in the four figures increase greatly compared with the figures above, no curves toward infinity are found in them. That means, the system is stable and the whole road section is not blocked.

4 Conclusions

In this paper, we adopt the variable substitution of original traffic flow models to convert traffic flow problems into system stability problems, just as the stability of discrete systems in the unit circle is expanded to the whole complex plane. Thus we can carry out the stability analysis directly by traffic flow models. Using the phase plane diagrams we can also describe all kinds of nonlinear phenomena observed in traffic flow. This will provide a theoretical basis for traffic control and decision. In order to specify this new method, we first build a new traffic flow model though substituting the variable in the GK model. Then the relationship between traffic congestion and the stability of the system can be obtained. So we can determine whether there will be traffic congestion or other abnormal phenomena from a global stability point. There is less study on the phenomena raised by different input and output on ramps. We use the new method to describe some of them with fixed vehicle generation rate but increasing initial homogeneous density with a single ramp. The results are also consistent with the diverse nonlinear dynamics phenomena observed in realistic traffic flow. Furthermore, the phase plane diagrams adopt the new model highlights the instability of the system. When the traffic became slow and congested, some curves tending to infinite can be seen from the phase plane diagrams as they account for a large proportion in the graphs while most small amplitude density fluctuations under stable traffic conditions just centered in a small area near the initial value.

In the future work, we will apply some mathematical tools such as branch and bound to analyze the nonlinear stability. It may be possible to apply some relative approaches of control theory to regulate the stability of traffic system through the equivalence relation between the traffic phenomena and system stability introduced in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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