A Sound and Complete Axiomatic System for Modality $\phi \equiv \_\_1\phi \wedge \_\_2\phi$

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Abstract: An axiomatic system is presented in this paper, which has a modal operator $\Box$ such that $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$, where $\Box_1$ and $\Box_2$ are the modal operators of the language for the axiom system $S5$. The axiomatic system for $\Box$ is proved to be sound and complete.
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1 Introduction

The modal logic has many axiomatic systems, such as $K, T, D, B, S4$ and $S5$ ([1]). The axiom system $S5$ is characterized by all equivalence frames([1]). The approximation spaces for Rough sets can be used as the possible-world semantics for $S5$. Let an approximate space (U, R) be an equivalence frame $\langle W, R\rangle$ for $S5$, i.e., $U = W$. Then for any formula φ, if the interpretation of φ corresponds to a subset X of U, then the lower and upper approximations of X correspond to the interpretations of $\Box\varphi$ and $\Diamond\varphi$, respectively, and the equivalence relation R corresponds to the accessibility relation for $\Box$([2]).

Given two approximation spaces (U, R_1) and (U, R_2), $R_1 \cup R_2$ may not be an equivalence relation. Given two modal operators $\Box_1$ and $\Box_2$, let R_1 and R_2 be the accessibility relations for $\Box_1$ and $\Box_2$, respectively. Let $\Box$ be a modal operator such that $R_1 \cup R_2$ is the accessibility relation for $\Box$, that is, $M, w \models \Box\varphi$ iff for any $w' \in W$ if $(w, w') \in R_1 \cup R_2$ then $M, w' \models \varphi$, which implies and is implied by that for any $w' \in W$ if $(w, w') \in R_1$ then $M, w' \models \varphi$ and for any $w' \in W$ if $(w, w') \in R_2$ then $M, w' \models \varphi$, i.e., $M, w \models \Box_1\varphi$ and $M, w \models \Box_2\varphi$ if and only if $M, w \models \Box\varphi$.

Let $\Box$ be a modal operator such that for any possible world $w, M, w \vDash \Box\varphi$ iff $M, w \vDash \Box_1\varphi$ and $M, w \vDash \Box_2\varphi$, i.e., for any formula φ, $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$. In this paper, we consider the modal operator $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$. We shall give the language, the syntax and the semantics for the modal logic with modal operator $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$. The axiomatic system for $\Box$ will be given and proved to be sound and complete.

The main contribution of this paper is that a propositional modal logic with a modal operator $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$. The axiomatic system for $\Box$ is sound, and complete with respect to the class of all reflective and symmetric frames, where the accessibility relation R for $\Box$ is equivalent to $R_1 \cup R_2$, where R_i is the equivalence relation for $\Box_i$ and $i = 1, 2$.

If $R_1 = R_2$, then the axiomatic system for $\Box$ turns out to be $S5$.

This paper is organized as follows: the propositional modal logic with modal operator $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$ is described in section 2, including the language, the syntax and the semantics for the logic. Then we shall give the axiomatic system for $\Box$ and prove the soundness theorem and the completeness theorem. Section 3 summaries results of the paper and discusses some possible extension of the logic.

2 The propositional modal logic with modal operator $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$

In this section, we shall give the language, the syntax and the semantics for the propositional modal logic with the modality $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$. The axiomatic system for $\Box$ is denoted by ${}^{S5_1}\wedge^{S5_2}$, then we prove that ${}^{S5_1}\wedge^{S5_2}$ is sound and complete.

2.1 The language, syntax and semantics for the logic

The language for ${}^{S5_1}\wedge^{S5_2}$ contains the following symbols:
- propositional variables: $p_0, p_1, ...$;
- logical connectives: $\neg, \rightarrow$;
- modalities: $\Box, \Box_1, \Box_2$;
- auxiliary symbols: (,).

Formulas:

$$\varphi := p|\varphi_1 \rightarrow \varphi_2|\neg\varphi_1|\Box\varphi_1;$$
$$\Box\varphi := \Box_1\varphi_1 \wedge \Box_2\varphi_1.$$

Other operators:

$$(\alpha \vee \beta) =_{\text{def}} (\neg\alpha \rightarrow \beta);$$
$$(\alpha \wedge \beta) =_{\text{def}} \neg(\alpha \rightarrow \neg\beta);$$
$$(\alpha \leftrightarrow \beta) =_{\text{def}} ((\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha));$$
$$(\Diamond\alpha) =_{\text{def}} (\neg\Box\neg\alpha).$$

Definition 2.1. A frame F is a triple $\langle W, R_1, R_2\rangle$, where W is a non-empty set of possible worlds, and $R_1 \subseteq W^2$ and $R_2 \subseteq W^2$ are the equivalence relations defined over the members of W and the accessibility relations for $\Box_1$ and $\Box_2$, respectively.

Definition 2.2. A model M is a quadruple $\langle W, R_1, R_2, I\rangle$, where $\langle W, R_1, R_2\rangle$ is a frame and I is an interpretation such that for any propositional variable p $I(p) \subseteq W$ and for any $w \in I(p)$ p is true in w.

A satisfaction relation $\vDash$, between any formula φ and any possible world w, is defined as follows:

Definition 2.3. Given any model M, any possible world $w \in W$ and any formula φ,

$$M, w \vDash \varphi \text{ iff } \begin{cases} w \in I(p) & \text{if } \varphi = p \\ M, w \nvDash \varphi_1 & \text{if } \varphi = \neg\varphi_1 \\ M, w \vDash \varphi_1 \Rightarrow M, w \models \varphi_2 & \text{if } \varphi = \varphi_1 \to \varphi_2 \\ \text{for all } w' \in W \text{ if } wR_1w' \text{ then } M, w' \vDash \varphi_1, \text{ and} & \\ \quad \text{for all } w' \in W \text{ if } wR_2w' \text{ then } M, w' \vDash \varphi_1 & \text{if } \varphi = \Box\varphi_1 \end{cases}$$

By the definition of the satisfaction relation, we can give the following definition:

Definition 2.4. A formula φ is valid in a model M, dented by $M \vDash \varphi$, iff for any $w \in W$ $M, w \vDash \varphi$; a formula φ is valid in a frame F, denoted by $F \vDash \varphi$, iff for any model M based on F $M \vDash \varphi$; let C be a class of frames. A formula φ is valid in C iff for any $F \in C$ $F \vDash \varphi$; $\Sigma \vDash_C \varphi$ iff for any frame $F \in C$ if $F \vDash \Sigma$ then $F \vDash \varphi$. If $\Sigma = \emptyset$ then $\vDash_C \varphi$.

Now we give the following axiom schemas and inference rules for ${}^{S5_1}\wedge^{S5_2}$:

- **Axiom schemes:**

$L1$	$\varphi \to (\psi \to \varphi)$
$L2$	$(\varphi \to (\psi \to \mu)) \to ((\varphi \to \psi) \to (\varphi \to \mu))$
$L3$	$(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$
$L4$	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$L5$	$\Box\varphi \to \varphi$
$L6$	$\varphi \to \Box\Diamond\varphi$
$L7_1$	$\Box_1\varphi \to \Box_1\Box_1\varphi$
$L7_2$	$\Box_2\varphi \to \Box_2\Box_2\varphi$

- **Inference rules:**

$$(MP) \quad \frac{\varphi, \varphi \to \psi}{\psi}$$

$$(N) \quad \frac{\varphi}{\Box\varphi}$$

Definition 2.5. A formula φ is provable from Γ, denoted by $\Gamma \vdash \varphi$, if there is a sequence of formulas $\varphi_1, ..., \varphi_n$ such that $\varphi = \varphi_n$, and for each $1 \leq i \leq n$, either φ_i is an axiom or a formula in Γ, or is deduced from the previous formulas via one of the deduction rules.

2.2 The Soundness Theorem

This section is to prove the soundness theorem by induction on the length of proofs. Before giving the proof, we give the following lemmas:

Lemma2.1. Each axiom schema is valid.

Proof. As for the axiom schema $L1, L2, L3$, we do not check their validity and two references are [1] and [3].

($L5$) By the definition 2.3, it is easy to prove it.

($L6$) By the definition 2.3, it is easy to prove it.

($L7_1$) Since the accessibility relation R_1 for $\Box_1$ is an equivalence relation, it follows that $\Box_1\varphi \to \Box_1\Box_1\varphi$ is valid.

($L7_2$) Since the accessibility relation R_2 for $\Box_2$ is an equivalence relation, it follows that $\Box_2\varphi \to \Box_2\Box_2\varphi$ is valid.

$\Box$

Lemma2.2. The deduction rules preserve validity.

Proof. We prove that (N) preserves the validity. Since for any model $\langle W, R_1, R_2, I\rangle$ based on any frame $\langle W, R_1, R_2\rangle$ and any $w \in W$, $M, w \models \varphi$.

Let w_1 be any possible world. Since R_1 and R_2 are the equivalence relations on W, we can obtain that for any $w_1' \in W$ if $w_1 R_1 w_1'$ then $M, w_1' \models \varphi$, and for any $w_1'' \in W$ if $w_1 R_2 w_1''$ then $M, w_1'' \vDash \varphi$. It follows that $M, w_1 \vDash \Box\varphi$. Since for any $w_1 \in W$ $M, w_1 \vDash \Box\varphi$, it follows that $\vDash \Box\varphi$.

$\Box$

Theorem 2.1(The Soundness Theorem). For any set of formulas Γ and formula φ, if $\Gamma \vdash \varphi$, then $\Gamma \vDash \varphi$.

Proof. For any set of formulas Γ and formula φ, since $\Gamma \vdash \varphi$, φ is the last member of a sequence which is a deduction from Γ. So we can use induction on the number of the sequence to prove this theorem as follows:

For the base step, the sequence has only one formula, namely φ. Then φ must be an axiom of ${}^{S5_1}\wedge^{S5_2}$ or a member of Γ, and then $\Gamma \models \varphi$.

Now suppose that the sequence contains n formulas,where n>1,and suppose as induction hypothesis that $\Gamma \models \alpha$ follows from $\Gamma \vdash \alpha$,which sequence is fewer than n members. There are the following cases:

Case a. φ is an axiom of ${}^{S5_1}\wedge^{S5_2}$ or a member of Γ, then we have $\Gamma \models \varphi$

Case b. φ is obtained by modus ponens rule from a formula ψ and a formula $\psi \to \varphi$ in the sequence. So by induction hypothesis, it obtains that $\Gamma \models \psi$ and $\Gamma \models \psi \to \varphi$, then it follows that $\Gamma \models \varphi$.

Case c. $\varphi = \Box\psi$ is obtained by the inference rule N from ψ. So by induction hypothesis, it follows that $\Gamma \models \psi$. Since if for any model $\langle W, R_1, R_2, I\rangle$

based on any frame $\langle W, R_1, R_2\rangle$ and any $w \in W$ $M, w \vDash \psi$ then for any model $\langle W, R_1, R_2, I\rangle$ based on any frame $\langle W, R_1, R_2\rangle$ and any $w \in W$ $M, w \vDash \Box\psi$, it follows that $\Gamma \models \Box\psi$.

$\Box$

2.3 The completeness theorem

The completeness theorem is to be proved in this section. The proof method of the complete theorem is similar to the classical canonical model method ([1]). We shall construct two relations on W and prove whether the two relations are equivalence relations or not. And one relation corresponds to the accessibility relation for $\Box_1$, while the other corresponds to the accessibility relation for $\Box_2$.

Definition 2.6. Γ is consistent iff there is no finite set $\{\varphi_1,...,\varphi_n\} \subseteq \Gamma$ such that:

$$\vdash \neg(\varphi_1 \wedge ... \wedge \varphi_n).$$

By definition 2.6, we can prove that Γ is inconsistent iff there is some formula φ such that $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.

Lemma 2.3. Suppose that Σ is a consistent set of formulas. Then there is a maximal consistent set of formulas Σ^* such that $\Sigma \subseteq \Sigma^*$.

In constructing a model in which the possible worlds are maximal consistent sets of formulas we will have to specify when one world is accessible from another(that model, in this paper, is also called canonical model). Thereby, the accessibility relation R_1 and R_2, in the canonical model, is defined as follows:

Definition 2.7. For any two distinct maximal consistent sets Σ_1^*,Σ_2^*, we define two binary relations R_1 and R_2 on W as follows:

(1) We shall say that $\Sigma_1^* R_1 \Sigma_2^*$ iff Σ_1^* and Σ_2^* satisfy the following condition:

For any formula φ if $\Box\varphi \in \Sigma_1^*$ then $\varphi \in \Sigma_2^*$ (written:$S^-(\Sigma_1^*) = \{\varphi : \Box\varphi \in \Sigma_1^*$ $\}$).

(2) We shall say that $\Sigma_1^* R_2 \Sigma_2^*$ iff Σ_1^* and Σ_2^* satisfy the following condition:

For any formula φ if $\Box\varphi \in \Sigma_1^*$ then $\Box\varphi \in \Sigma_2^*$ (written: $S(\Sigma_1^*) = \{\Box\varphi : \Box\varphi \in \Sigma_1^*$ $\}$).

Lemma 2.4. Let $\Gamma^* = \{\Sigma_0^*, \Sigma_1^*, ...\}$ be the set of all maximal consistent sets. If for any $i, j \in \mathrm{N}$ we define $\Sigma_i^* R_1 \Sigma_j^*$ iff $S^-(\Sigma_i^*) \subseteq \Sigma_j^*$, and $\Sigma_i^* R_2 \Sigma_j^*$ iff $S(\Sigma_i^*) \subseteq \Sigma_j^*$, then both the relation R_1 and R_2 are equivalence relations on W.

Proof. (1) In order to prove that R_1 is an equivalence relation, we shall prove the following three conditions:

1) For any $i \in N$, if $\Sigma_i^* \in \Gamma^*$ then $\Sigma_i^* R_1 \Sigma_i^*$.

2) For any $\Sigma_i^*, \Sigma_j^* \in \Gamma^*$, if $\Sigma_i^* R_1 \Sigma_j^*$ then $\Sigma_j^* R_1 \Sigma_i^*$.

3) For any $\Sigma_1^*, \Sigma_2^*, \Sigma_3^* \in \Gamma^*$, if $\Sigma_1^* R_1 \Sigma_2^*$ and $\Sigma_2^* R_1 \Sigma_3^*$ then $\Sigma_1^* R_1 \Sigma_3^*$.

1). For any $i \in N$, we shall prove $S^-(\Sigma_i^*) \subseteq \Sigma_i^*$. For any formula φ, if $\Box\varphi \in \Sigma_i^*$ then $\varphi \in S^-(\Sigma_i^*)$. Since $\Box\varphi \to \varphi(L5)$ and $\Box\varphi \in \Sigma_i^*$, $\varphi \in \Sigma_i^*$. It follows that $S^-(\Sigma_i^*) \subseteq \Sigma_i^*$.

2). We shall prove that if $S^-(\Sigma_i^*) \subseteq \Sigma_j^*$ then $S^-(\Sigma_j^*) \subseteq \Sigma_i^*$, that is to say, we shall prove that for any formula β if $\Box\beta \in \Sigma_j^*$ then $\beta \in \Sigma_i^*$. Suppose $\beta \notin \Sigma_i^*$. $\neg\beta \in \Sigma_i^*$. By $L6$ and $\neg\beta \in \Sigma_i^*$ it follows that $\Box\Diamond\neg\beta \in \Sigma_i^*$.

Since $S^-(\Sigma_i^*) \subseteq \Sigma_j^*$, it follows that $\Diamond\neg\beta \in \Sigma_j^*$, that is, $\neg\Box\beta \in \Sigma_j^*$. Since $\neg\Box\beta \in \Sigma_j^*$ and $\Box\beta \in \Sigma_j^*$, Σ_j^* is not consistent, which is a contradiction to the hypothesis of this lemma.

3). We need to prove that if $S^-(\Sigma_1^*) \subseteq \Sigma_2^*$ and $S^-(\Sigma_2^*) \subseteq \Sigma_3^*$ then $S^-(\Sigma_1^*) \subseteq \Sigma_3^*$, that is to say, for any formula β, if $\Box\beta \in \Sigma_1^*$ then $\beta \in \Sigma_3^*$. We can prove it by $L5$ and $L7_1$.

What we need to explain is that R_1 is the accessibility relation for $\Box_1$ and in this case for any formula φ $\Box\varphi \equiv \Box_1\varphi$. Thereby, we can use the axiom schema $L7_1$.

(2) Now, we prove that the relation R_2 is an equivalence relation. the following three conditions shall be proved:

1) For any $i \in N$, if $\Sigma_i^* \in \Gamma^*$ then $\Sigma_i^* R_2 \Sigma_i^*$.

2) For any $\Sigma_i^*, \Sigma_j^* \in \Gamma^*$, if $\Sigma_i^* R_2 \Sigma_j^*$ then $\Sigma_j^* R_2 \Sigma_i^*$.

3) For any $\Sigma_1^*, \Sigma_2^*, \Sigma_3^* \in \Gamma^*$, if $\Sigma_1^* R_2 \Sigma_2^*$ and $\Sigma_2^* R_2 \Sigma_3^*$ then $\Sigma_1^* R_2 \Sigma_3^*$.

It is easy to prove the item 1)-3). We omit the proof procedures.

$\Box$

Lemma 2.5. Let Γ be any consistent set of formulas containing $\neg\Box\psi$, then $S^-(\Gamma) \cup \{\neg\psi\}$ is consistent and $S(\Gamma) \cup \{\neg\psi\}$ is consistent, where $S^-(\Gamma) = \{\varphi : \Box\varphi \in \Gamma\ \}$ and $S(\Gamma) = \{\Box\varphi : \Box\varphi \in \Gamma\ \}$.

Proof. (1) We shall prove that $S^-(\Gamma) \cup \{\neg\psi\}$ is consistent as follows:

Suppose that $S^-(\Gamma) \cup \{\neg\psi\}$ is not consistent. Then there exists some finite subset $\{\varphi_1, ..., \varphi_n\} \cup \{\neg\psi\}$ of $S^-(\Gamma) \cup \{\neg\psi\}$ such that $\vdash \neg(\varphi_1 \wedge ... \wedge \varphi_n \wedge \neg\psi)$.

Then

$$\begin{aligned} \vdash (\varphi_1 \wedge ... \wedge \varphi_n) \to \neg\psi \quad & \text{iff} \vdash \Box(\varphi_1 \wedge ... \wedge \varphi_n) \to \psi \\ & \text{iff} \vdash \Box(\varphi_1 \wedge ... \wedge \varphi_n) \to \Box\psi \\ & \text{iff} \vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \to \Box\psi \\ & \text{iff} \vdash \neg(\Box\varphi_1 \wedge ... \wedge \Box\varphi_n \wedge \neg\Box\psi) \end{aligned}$$

Thereby, $\{\Box\varphi_1, ..., \Box\varphi_n\} \cup \{\neg\Box\psi\}$ is not consistent. By the definition of $S^-(\Gamma)$ it follows that $\{\Box\varphi_1, ..., \Box\varphi_n\} \cup \{\neg\Box\psi\}$ is a subset of Γ. Thereby, Γ is not consistent, which is a contradiction to the hypothesis of this lemma.

(2) We shall prove that $S(\Gamma) \cup \{\neg\psi\}$ is consistent as follows. Suppose that $S(\Gamma) \cup \{\neg\psi\}$ is not consistent. Then there exists some finite subset $\{\Box\varphi_1, ..., \Box\varphi_n\} \cup \{\neg\psi\}$ of $S(\Gamma) \cup \{\neg\psi\}$ such that $\vdash \neg(\Box\varphi_1 \wedge ... \wedge \Box\varphi_n \wedge \neg\psi)$. Then

$$\begin{aligned} \vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \to \neg\psi \quad & \text{iff } \vdash \Box(\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \to \psi \\ & \text{iff } \vdash \Box(\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \to \Box\psi \\ & \text{iff } \vdash (\Box\varphi_1 \wedge ... \wedge \Box\varphi_n) \to \Box\psi \\ & \text{iff } \vdash \neg(\Box\Box\varphi_1 \wedge ... \wedge \Box\Box\varphi_n \wedge \neg\Box\psi) \end{aligned}$$

Thereby, $\{\Box\Box\varphi_1, ..., \Box\Box\varphi_n\} \cup \{\neg\Box\psi\}$ is not consistent. For any $\Box\varphi_i$, by $L7_2(\Box\varphi \equiv \Box_2\varphi)$ it follows that $\Box\Box\varphi \in \Gamma$, where i=1,...,n. Thereby, $\{\Box\Box\varphi_1, ..., \Box\Box\varphi_n\} \cup \{\neg\Box\psi\}$ is a subset of Γ. Then Γ is not consistent, which is a contradiction to the hypothesis of this lemma.

What we need to explain is that R_2 is the accessibility relation for $\Box_2$ and in this case for any formula φ $\Box\varphi \equiv \Box_2\varphi$. Thereby, we can use the axiom schema $L7_2$.

$\Box$

The canonical model for ${}^{S5_1}\wedge^{S5_2}$, M, is like any other model, a quadruple $\langle W, R_1, R_2, I\rangle$. W is the set of all sets of maximal consistent sets of formulas. I.e. $w \in W$ iff w is a maximal consistent set of formulas. If w and w' are both in W, then wR_1w' iff $S^-(w) \subseteq w'$. And if w and w' are both in W, then wR_2w' iff $S(w) \subseteq w'$. For any propositional variable p $I(p) \subseteq W$, and for any $w \in I(p)$ p is true in w iff $p \in w$. For any other formula this has to be proved as follows:

Lemma 2.6. Let $M = \langle W, R_1, R_2, I\rangle$ be the canonical model for ${}^{S5_1}\wedge^{S5_2}$. Then for any formula φ and any world w, $M, w \vDash \varphi$ iff $\varphi \in w$.

Proof. We prove the lemma by induction on the structure of formulas.

Case a. $\varphi := p$: By definition, this lemma holds.

Case b. $\varphi := \neg\alpha$:

$$\begin{aligned} M, w \vDash \neg\alpha \quad & \text{iff } M, w \nvDash \alpha \\ & \text{iff } \alpha \notin w \\ & \text{iff } \neg\alpha \in w \end{aligned}$$

Case c. $\varphi := \alpha \to \beta$:

$$\begin{aligned} \alpha \to \beta \notin w \quad & \text{iff } \neg(\alpha \to \beta) \in w \\ & \text{iff } \alpha \in w \text{ and } \neg\beta \in w \\ & \text{iff } M, w \vDash \alpha \text{ and } M, w \nvDash \beta \\ & \text{iff } M, w \nvDash \alpha \to \beta \end{aligned}$$

Case d. $\varphi := \Box\alpha$:

($\Leftarrow$) **Subcase d.1**. $\Box\alpha \in w$: By the definition of R_1, R_2, we have the following two cases:

(1) For any $w' \in W$, if wR_1w' then $\alpha \in w'$. $\alpha \in w'$ iff $M, w' \vDash \alpha$ by induction hypothesis. Then $M, w \vDash \Box\alpha$ because for any w' if wR_1w' and $M, w' \vDash \alpha$.

(2) For any $w' \in W$, if wR_2w' then $\Box\alpha \in w'$. Since $\Box\alpha \in w'$ and $\Box\alpha \to \alpha$, $\alpha \in w'$. $\alpha \in w'$ iff $M, w' \vDash \alpha$ by induction hypothesis. Then $M, w \vDash \Box\alpha$ because for any w' if wR_2w' and $M, w' \vDash \alpha$.

Subcase d.2. $\neg\Box\alpha \in w$: By the lemma 2.5 , both $S^-(w) \cup \{\neg\alpha\}$ and $S(w) \cup \{\neg\alpha\}$ are consistent. By the lemma 2.3, we can enlarge $S^-(w) \cup \{\neg\alpha\}$ and $S(w) \cup \{\neg\alpha\}$ into maximal consistent sets of formulas w_1 and w_2, respectively. Thereby, there exist w_1, w_2 such that $S^-(w) \subseteq w_1$ and $S(w) \subseteq w_2$.

Since $S^{-}(w) \subseteq w_1$ and $S(w) \subseteq w_2$, by the definition of R_1 and R_2, it follows that wR_1w_1 and wR_2w_2. For $\neg\alpha \in w_1$ and $\neg\alpha \in w_2$, it follows that $M, w_1 \models \neg\alpha$ and $M, w_2 \models \neg\alpha$ by induction hypothesis. So there exist w_1, w_2 such that wR_1w_1 and wR_2w_2 and $M, w_1 \models \neg\alpha$ and $M, w_2 \models \neg\alpha$. Therefore, by the definition 2.3, it follows that $M, w \nvDash \Box\alpha$.

($\Rightarrow$) **Subcase d.3**. $M, w \vDash \Box\alpha$: By the definition 2.3, we have:

(1) For any $w' \in W$, if wR_1w' then $M, w' \vDash \alpha$; and

(2) For any $w' \in W$, if wR_2w' then $M, w' \vDash \alpha$.

From (1), we have:

(3) For any $w' \in W$, if wR_1w' then $\alpha \in w'$ by induction hypothesis.

From (2),we have:

(4) For any $w' \in W$, if wR_2w' then $\alpha \in w'$ by induction hypothesis.

Assume $\neg\Box\alpha \in w$. Since w is a maximal consistent set of formulas, it follows that there exists $w_1, w_2 \in W$ such that wR_1w_1 and wR_2w_2 and $\neg\alpha \in w_1$ and$\neg\alpha \in w_2$ by the lemma 2.5. So $\neg\alpha \in w_1$ is a contradiction to (3), and $\neg\alpha \in w_1$ and $\neg\alpha \in w_2$ is also a contradiction to (4). Thereby, $\Box\alpha \in w$ for w is a maximal consistent set of formulas.

Subcase d.4. $M, w \nvDash \Box\alpha$: By the definition 2.3, we have:

(1) There exists $w_1 \in W$ such that wR_1w_1 and $M, w_1 \nvDash \alpha$; or

(2) There exists $w_2 \in W$ such that wR_2w_2 and $M, w_2 \nvDash \alpha$.

From (1), by induction hypothesis, $\neg\alpha \in w_1$; and from (2), $\neg\alpha \in w_2$. Suppose $\Box\alpha \in w$, by the definition of R_1 and R_2, it follows that:

(3) For any $w' \in W$ if wR_1w' then $\alpha \in w'$; and

(4) For any $w' \in W$ if wR_2w' then $\Box\alpha \in w'$.

From (4) and $L5$, we have:

(5) For any $w' \in W$ if wR_2w' then $\alpha \in w'$.

It follows that (3) is in contradiction with wR_1w_1 and $\neg\alpha \in w_1$. And (5) also is in contradiction with wR_2w_2 and $\alpha \in w_2$. Thereby, $\neg\Box\alpha \in w$ for that w is a maximal consistent set of formulas.

$\Box$

Theorem 2.2(The Completeness Theorem): For any set of formulas Γ and formula φ, if $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

Proof. Suppose $\Gamma \nvdash \varphi$, so $\Gamma \cup \{\neg\varphi\}$ is consistent. Then there is a maximal consistent set of formulas w such that $\Gamma \cup \{\neg\varphi\} \subseteq w$ by the lemma 2.3.

Let $M = \langle W, R_1, R_2, I\rangle$ be a canonical model, where W is the set of all sets of maximal consistent sets of formulas. I.e. $w \in W$ iff w is a maximal consistent set of formulas. If w and w' are both in W, then wR_1w' iff $S^{-}(w) \subseteq w'$. And if w and w' are both in W, then wR_2w' iff $S(w) \subseteq w'$. By the lemma 2.4 it follows that R_1 and R_2 are the equivalence relations on W.

For any formula $\alpha \in \Gamma \cup \{\neg\varphi\}$, $\alpha \in w$. By the theorem 2.6, $M, w \vDash \alpha$. So $M, w \vDash \Sigma$ and $v \vDash \neg\varphi$, which is a contradiction to $\Sigma \vDash \varphi$.

So for any set of formulas Γ and formula φ, if $\Gamma \vDash \varphi$, then $\Gamma \vdash \varphi$.

$\Box$

3 Conclusion

In this paper, we present an axiomatic system for a modality $\Box\varphi \equiv \Box_1\varphi \wedge \Box_2\varphi$, and prove that the axiomatic system for $\Box$ is sound and complete. The axiomatic system for $\Box$ is different from $S5$. What we need to point out is that $L7_1$ and $L7_2$ are not the axiom schemas for $\Box$, which is needed when proving the completeness theorem. An interesting problem is to give a sound and complete axiomatic system for the modality corresponding to the accessibility relation $R = R_1 \cap R_2$, where the equivalence relation R_i is the accessibility relation for $\Box_i, i = 1, 2$.

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