A Lower Bound on Load of Coded Caching Schemes for Finite Subpacketizations

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***Abstract*—Coded caching is a technique to create coded multicast opportunities for cache-aided networks. In coded caching problem, a fundamental but open question is: what is the minimum transmission load given any fixed subpacketization level? In this paper, we propose a lower bound on the transmission load for any fixed subpacketization by studying the combinatorial structure of corresponding placement delivery array, which was introduced by Yan et al. to reformulate the centralized coded caching schemes. Then we show that some schemes generated by the well known scheme proposed by Maddah-Ali and Niesen (MN), and some scheme generated by Packing (a classic concept of combinatorial design theory), can achieve our lower bound. This implies that our lower bound is tight for some cases.**

***Index Terms*—Code caching, placement delivery array, Packing**

I. Introduction

Caching has been recognized as an effective method to smooth out network traffic during peak times by the following idea: In cache-aided networks, some content is proactively stored into the users' local cache memories during off-peak hours in the hope that the pre-stored content will be required during peak hours. When this happens, content is retrieved locally thereby reducing the communication load from the server to the users. Coded caching was introduced by Maddah-Ali and Niesen (MN) in [1] to further reduce the amount of transmission by creating broadcast coding opportunities, where the central server transmits some coded symbols and each user uses their cache to cancel the non-desired file, and has been widely used in heterogeneous wireless networks.

A. Coded caching system

In a (K, M, N) caching system, a single server containing N independent files with the same length connects to K users over a shared link and each user has a cache memory of size M files with $N \geq \max\{K, M\}$. Denote the N files by $\mathcal{W} = \{W_0, \ldots, W_{N-1}\}$ and K users by $\mathcal{K} = \{0, \ldots, K-1\}$. An F-division (K, M, N) coded caching scheme consists of two phases as follows:

- **Placement phase:** During the off peak traffic time, each file is divided into F equal packets. Then each user caches some packets or linear combinations of packets from the server. If packets are cached directly, it is called uncoded placement; if linear combinations of packets are cached, we call it coded placement. In this phase we assume that the server does not know the users' later requests.
- **Delivery phase:** During the peak traffic times, each user randomly requests one file from the files set $\mathcal{W}$ independently. The request vector is denoted by $\mathbf{d} = (d_0, \cdots, d_{K-1})$, i.e., user k requests the d_k-th file W_{d_k}, where $d_k \in \{0, \ldots, N-1\}$ and $k \in \mathcal{K}$. The server broadcasts a coded signal (linear combination of some required packets) of size $S_{\mathbf{d}}$ packets to users such that each user is able to recover its requested file with the help of its cache contents.

The transmission load, denoted by R, of a coded caching scheme is defined as the maximal normalized transmission amount among all the requests in the delivery phase. In practice the file size can be any finite positive number. This implies that the subpacketization is finite. So we prefer to design a scheme with the transmission load as small as possible for any fixed subpacketization.

B. Prior Work

The first well known scheme called MN scheme was proposed in [1] whose communication load is optimal under the constraint of uncoded placement and $N \geq K$ [2], and generally order optimal within a factor of 2 [3]. However there exists a main practical issue of the MN scheme, i.e., its subpacketization increases exponentially with the number of users. In order to study the finite subpacketization, a matrix with F rows and K columns called placement delivery array (PDA), which can be used to realize a coded caching under uncoded caching placement, was proposed in [4]. There are other viewpoints of characterizing coded caching schemes from the view point of graph theory, combinatorial design theory, and coding theory [5]–[8] under uncoded caching placement. The authors in [9] introduced linear characterization of coded caching schemes with coded caching placement from the view point of linear algebra. Clearly all the schemes under uncoded caching placement can be regarded as special case of linear characterization. From linear characterization, the problem of designing schemes is transformed to constructing three classes of matrices satisfying certain rank conditions which represent user caching strategy, server broadcasting strategy and user decoding strategy. Furthermore by these three classes of matrices the authors showed that the minimum storage regenerating codes with optimal repairing bandwidth proposed by Dimakis et al in [10] can be used to generate the related linear coded caching scheme. That is the reason why there

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are many similar structures used between the constructions of coded caching schemes and the minimum storage regenerating codes. Since the study of linear coded caching scheme is much more complicated than that of schemes under uncoded caching placement, there is only one class of linear coded caching scheme proposed in [9], which has better performance than the schemes under uncoded caching placement for the same subpacketization and memory ratio, while there are many constructions of the schemes under uncoded caching placement.

In fact all the characterizations in [5]–[8] can be represented by PDAs. Very recently the authors in [11] proposed a unified constructing framework for PDA which can represent all the constructions in [4]–[6], [8] and son on. As a result the problem of designing a PDA can be transformed into a problem of choosing a row index matrix and a column index set appropriately.

C. Contribution and organization

In this paper we focus on coded caching problems under uncoded caching placement, and study the minimum transmission load for any fixed user number, memory ratio and subpacketization. Firstly we propose a lower bound on the transmission load for any coded caching scheme which can be realized by an appropriate PDA. Then taking the MN scheme and its conjugate and some scheme with special user number, memory ratio and subpacketizations, we show our lower bound is tight.

The rest of this paper is organized as follows. In Section II, the concept of PDA and the scheme realized by it are introduced. In Section III a lower bound on the transmission load for any user number, memory ratio and subpacketization is derived. Finally, Section IV concludes the paper.

II. Placement delivery array

In this paper we will use bold capital letters, bold lower case letters and curlicue letters to denote arrays, vectors and sets respectively. For any positive integers a, let $[a] = \{0, 1, \ldots, a-1\}$. Now let us see the following definition of PDA.

Definition 1. *([4]) For positive integers K, F, and S, an $F \times K$ array $\mathbf{P} = (p_{j,k})$, $j \in [F], k \in [K]$, composed of a specific symbol "$*$" called star and S symbols in $[S]$, is called a (K, F, S) placement delivery array (PDA) if it satisfies the following condition.*

C1. *For any two distinct entries p_{j_1,k_1} and p_{j_2,k_2}, $p_{j_1,k_1} = p_{j_2,k_2} = s$ is an integer only if*

a. *$j_1 \neq j_2$, $k_1 \neq k_2$, i.e., they lie in distinct rows and distinct columns; and*

b. *$p_{j_1,k_2} = p_{j_2,k_1} = *$, i.e., the corresponding 2×2 subarray formed by rows j_1, j_2 and columns k_1, k_2 must be of the following form*

$$\left(\begin{array}{cc} s & * \\ * & s \end{array}\right) \text{ or } \left(\begin{array}{cc} * & s \\ s & * \end{array}\right).$$

In addition, for any positive integer $Z \leq F$, the array $\mathbf{P}$ is denoted by (K, F, Z, S) PDA if the following condition is further satisfied:

C2. each column has exactly Z stars.

For instance, it is easy to verify that the array is a $(6, 4, 2, 4)$ PDA.

$$\mathbf{P} = \left(\begin{array}{cccccc} * & * & * & 0 & 1 & 2 \\ * & 0 & 1 & * & * & 3 \\ 0 & * & 2 & * & 3 & * \\ 1 & 2 & * & 3 & * & * \end{array}\right). \tag{1}$$

Yan et al. in [4] showed that any PDA can be used to generate a coded caching scheme. That is the following result.

Theorem 1. *([4]) If there exits a (K, F, Z, S) PDA, then by Algorithm 1 we can obtain an F-division (K, M, N) coded caching scheme with $\frac{M}{N} = \frac{Z}{F}$ and transmission load $R = \frac{S}{F}$.*

Algorithm 1 Coded caching scheme based on PDA in [4]

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1: procedure PLACEMENT(P, W)
2:     Split each file W_n ∈ W into F packets, i.e., W_n = {W_{n,j} | j ∈ [F]}.
3:     for k ∈ K do
4:         Z_k ← {W_{n,j} | p_{j,k} = *, ∀ n ∈ [N]}
5:     end for
6: end procedure
7: procedure DELIVERY(P, W, d)
8:     for s = 0, 1, ··· , S − 1 do
9:         Server sends ⊕_{p_{j,k}=s, j∈[F], k∈[K]} W_{d_k,j}.
10:    end for
11: end procedure
```

Let us take the following example to illustrate Algorithm 1.

Example 1. *Using the PDA in* (1) *and Algorithm 1, we can obtain a 4-division $(6, 3, 6)$ coded caching scheme as follows.*

- ***Placement Phase**: From Line 2 of Algorithm 1, we have $W_n = \{W_{n,0}, W_{n,1}, W_{n,2}, W_{n,3}\}$, $n \in [6]$. Then by Lines 3-5, the contents cached by user 0,1, ..., 5 are respectively*

$$\mathcal{Z}_0 = \{W_{n,0}, W_{n,1} | n \in [6]\}, \mathcal{Z}_1 = \{W_{n,0}, W_{n,2} | n \in [6]\},$$
$$\mathcal{Z}_2 = \{W_{n,0}, W_{n,3} | n \in [6]\}, \mathcal{Z}_3 = \{W_{n,1}, W_{n,2} | n \in [6]\},$$
$$\mathcal{Z}_4 = \{W_{n,1}, W_{n,3} | n \in [6]\}, \mathcal{Z}_5 = \{W_{n,2}, W_{n,3} | n \in [6]\}.$$

- ***Delivery Phase**: Assume that the request vector is $\mathbf{d} = (0, 1, 2, 3, 4, 5)$. By Lines 8-10, the signals sent by the server are listed in Table I.*

From the above introduction, in order to get the smallest transmission load for any fixed user number K, subpacketization F and memory ratio $M/N = Z/F$, we only need to construct a PDA with S as small as possible, and thus define

$$\mathcal{S}(K, F, Z) = \min_{(K,F,Z,S) \text{ PDA}} S. \tag{2}$$

Then, a (K, F, Z, S) PDA is said to be optimal if $S = \mathcal{S}(K, F, Z)$.

Here we list some PDAs which will be useful in this paper.

Lemma 1. *(MN PDA, [4]) MN scheme is equivalent to a $(K, \binom{K}{t}, \binom{K-1}{t-1}, \binom{K}{t+1})$ PDA with t stars in each row where $t = KM/N$ and transmission load $R = \frac{K-t}{t+1}$.*

We describe how to construct an MN PDA.

Construction 1. *([1], [4]) For each integer $t \in [K]$, we let $F = \binom{K}{t}$. Arrange all the subsets with size $t+1$ of $[K]$ in the lexicographic order and define $f_{t+1}(\Omega)$ to be its order minus 1 for any subset Ω of size $t+1$. Clearly, f_{t+1} is a bijection from $\{\Omega \subset [K] : |\Omega| = t+1\}$ to $[\binom{K}{t+1}]$. Then, an MN PDA is defined as a $\binom{K}{t} \times K$ array $\mathbf{P} = (p_{\mathcal{T},k})_{\mathcal{T}\subset[K],|\mathcal{T}|=t,k\in[K]}$ by*

$$p_{\mathcal{T},k} = \begin{cases} f_{t+1}(\mathcal{T} \cup \{k\}), & \text{if } k \notin \mathcal{T} \\ *, & \text{otherwise} \end{cases} \tag{3}$$

where the rows are indexed by all the subsets $\mathcal{T} \subset [K]$ and $|\mathcal{T}| = t$.

Based on the MN PDA in Construction 1, we can define its conjugate $\mathbf{P}' = (p'_{k,\mathcal{T}})_{k\in[K],\mathcal{T}\subset[K],|\mathcal{T}|=t}$ where $p'_{k,\mathcal{T}} = p_{\mathcal{T},k}$. Let $F = K$ and $Z = t$ in Construction 1. Then the following result can be obtained.

Lemma 2. *([12], Conjugate MN PDA) For any positive integers F and Z with $0 < Z < F$, there exists a $(\binom{F}{Z}, F, Z, \binom{F}{Z+1})$ PDA $\mathbf{P}'$.*

For any positive integer m, it is easy to check that

$$\Big(\underbrace{\mathbf{P}', \mathbf{P}' + \binom{F}{Z+1}, \ldots, \mathbf{P}' + m\binom{F}{Z+1}}_{m}\Big)$$

is an $(m\binom{F}{Z}, F, Z, m\binom{F}{Z+1})$ PDA where $\mathbf{P}'$ is the conjugate MN PDA in Lemma 2. That is the following result.

Lemma 3. *(Group conjugate MN PDA) For any positive integers F, m and Z with $0 < Z < F$, there exists an $(m\binom{F}{Z}, F, Z, m\binom{F}{Z+1})$ PDA which leads to a (K, M, N) coded caching scheme with $\frac{M}{N} = \frac{Z}{F}$ and transmission load $R = m\frac{F-Z}{Z+1}$.*

III. Lower bounds on $\mathcal{S}(K, F, Z)$

In this section we will propose a lower bound on $\mathcal{S}(K, F, Z)$ and then show that for some parameters, our lower bound is tight.

In order to derive the lower bounds on $S(K, F, Z)$, the following notation is useful. Given a (K, F, Z, S) PDA $\mathbf{P} = (p_{j,k})$ where $j \in [F]$, $k \in [K]$, we can define the subsets $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_{K-1}$ of $[F]$ as follows.

$$\mathcal{A}_k = \{j \mid p_{j,k} \in [S], j \in [F]\}. \tag{4}$$

Here $\mathcal{A}_k$ denotes the index set of subfiles that are not cached by node $k \in [K]$. From above notation, the following statement holds.

Lemma 4. *Let $\mathcal{I}$ be the permutation set of $[K]$. For any (K, F, Z, S) PDA,*

$$S \geq \max\left\{ \sum_{h=0}^{K-1} \Big| \bigcap_{j=0}^{h} \mathcal{A}_{i_j} \Big| \;\Big|\; (i_0, \ldots, i_{K-1}) \in \mathcal{I} \right\} \tag{5}$$

where $\mathcal{A}_{i_j}$ is defined in (4).

Proof. According to (4), the sets $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_{K-1}$ can be obtained. For any permutation, say $(i_0, i_1, \ldots, i_{K-1}) \in \mathcal{I}$, we claim that for any two distinct $h, h' \in [K]$ and any two distinct $x_1 \in \bigcap_{j=0}^{h} \mathcal{A}_{i_j}$, $x_2 \in \bigcap_{j'=0}^{h'} \mathcal{A}_{i_j}$, the following two statements hold.

- $p_{x_1,i_h} \neq p_{x_2,i_h}$ and $p_{x_2,i_h} \neq p_{x_2,i_{h'}}$ holds by the first property in Definition 1.
- $p_{x_1,i_h} \neq p_{x_2,i_{h'}}$ by the second property in Definition 1. Otherwise a contradiction to the definition of PDA since the cell (x_2, i_h) or $(x_1, i_{h'})$ contains an integer.

From above discussions, we have that the number of distinct elements in cells (x_1, i_h), $x_1 \in \bigcap_{j=0}^{h} \mathcal{A}_{i_j}$, exactly equals

$$|\mathcal{A}_{i_0}| + |\mathcal{A}_{i_0} \bigcap \mathcal{A}_{i_1}| + \ldots + \left| \bigcap_{j=0}^{K-1} \mathcal{A}_{i_j} \right|.$$

Clearly $S \geq |\mathcal{A}_{i_0}| + |\mathcal{A}_{i_0} \bigcap \mathcal{A}_{i_1}| + \ldots + |\bigcap_{j=0}^{K-1} \mathcal{A}_{i_j}|$. Then the proof is complete. □

Let us take $K = 6$, $F = 4$ and $Z = 1$ to illustrate the inequality in (5). It is easy to check that the following array is a $(6, 4, 1, 11)$ PDA.

$$\mathbf{P}_{4\times 6} = \begin{pmatrix} 0 & 1 & 2 & * & 6 & 7 \\ 3 & 4 & * & 2 & 8 & 9 \\ 5 & * & 4 & 1 & 10 & * \\ * & 5 & 3 & 0 & * & 10 \end{pmatrix}.$$

According to (4), we have $\mathcal{A}_0 = \{0, 1, 2\}$, $\mathcal{A}_1 = \{0, 1, 3\}$, $\mathcal{A}_2 = \{0, 2, 3\}$, $\mathcal{A}_3 = \{1, 2, 3\}$, $\mathcal{A}_4 = \{0, 1, 2\}$ and $\mathcal{A}_5 = \{0, 1, 3\}$. By (5) we have

$$\begin{aligned} &\max\left\{ \sum_{h=0}^{5} \Big| \bigcap_{j=0}^{h} \mathcal{A}_{i_j} \Big| \;\Big|\; (i_0, \ldots, i_5) \in \mathcal{I} \right\} \\ =&|\mathcal{A}_0| + |\mathcal{A}_0 \cap \mathcal{A}_4| + |\mathcal{A}_0 \cap \mathcal{A}_4 \cap \mathcal{A}_1| + |\mathcal{A}_0 \cap \mathcal{A}_4 \cap \\ &\mathcal{A}_1 \cap \mathcal{A}_5| + |\mathcal{A}_0 \cap \mathcal{A}_4 \cap \mathcal{A}_1 \cap \mathcal{A}_5 \cap \mathcal{A}_2| \\ =&3 + 3 + 2 + 2 + 1 \\ =&11. \end{aligned}$$

For any positive integers F and Z with $Z \leq F$, let Ω be the set consisting of all the $(F - Z)$-subsets of $[F]$. From above discussions, the following statement holds.

Theorem 2. *For any positive integers F, K and Z with $Z \leq F$, the following inequality always holds.*

$$S(K,F,Z) \geq \min_{\mathcal{A}_k\in\Omega, k\in[K]} \max\Big\{ \sum_{h=0}^{K-1} |\bigcap_{j=0}^{h} \mathcal{A}_{i_j}| \;\Big|\; (i_0,\ldots,i_{K-1}) \in \mathcal{I}\Big\}. \tag{6}$$

In the following let us consider two extreme cases. First let us consider the case $K = m\binom{F}{F-Z}$. When each $(F-Z)$-size subset of $[F]$ occurs exactly m times in $\{\mathcal{A}_k \mid k \in [K]\}$, it is not difficult to check that

$$\begin{aligned}&\max\Big\{ \sum_{h=0}^{K-1} |\bigcap_{j=0}^{h} \mathcal{A}_{i_j}| \;\Big|\; (i_0,\ldots,i_{K-1}) \in \mathcal{I}\Big\}\\ =&m\left(\binom{F-1}{F-Z-1}+\binom{F-2}{F-Z-2}+\cdots+\binom{Z+1}{1}+\binom{Z}{0}\right)\\ =&m\left(\binom{F-1}{Z}+\binom{F-2}{Z}+\cdots+\binom{Z+1}{Z}+\binom{Z}{Z}\right)\\ =&m\binom{F}{Z+1}.\end{aligned}$$

By induction, we can prove that

$$\begin{aligned}&m\binom{F}{Z+1}\\ =&\min_{\mathcal{A}_k\in\Omega, k\in[K]} \max\Big\{ \sum_{h=0}^{K-1} |\bigcap_{j=0}^{h} \mathcal{A}_{i_j}| \mid (i_0,\ldots,i_{K-1}) \in \mathcal{I}\Big\}\\ \leq&S(K,F,Z).\end{aligned}$$

So we have that the Group conjugate MN PDA in Lemma 3 is optimal.

Now let us consider the second case, i.e., the intersection of any two subsets $\mathcal{A}_k$ and $\mathcal{A}_{k'}$ contains at most one element for any $k, k' \in [K]$. For any K subsets $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_{K-1}$ with $|\mathcal{A}_k| = (F-Z)$, $k \in [K]$, by the Pigeonhole Principle, there must exist an element occurring in at least $\lceil (F-Z)K/F \rceil$ subsets. So we have

$$\begin{aligned}&\max\Big\{ \sum_{h=0}^{K-1} |\bigcap_{j=0}^{h} \mathcal{A}_{i_j}| \mid (i_0,\ldots,i_{K-1}) \in \mathcal{I}\Big\}\\ \geq&\lceil (F-Z)K/F \rceil + (F-Z) - 1.\end{aligned} \tag{7}$$

When we can find K subsets $\mathcal{A}_0$, $\mathcal{A}_1$, …, $\mathcal{A}_{K-1}$ with $|\mathcal{A}_k| = F-Z$, satisfying that $|\mathcal{A}_k \bigcap \mathcal{A}_{k'}| \leq 1$ for any distinct $k, k' \in [K]$, then the value in (7) can be obtained. In fact the concept of packing, whose definition is listed in the following, exactly satisfies this condition.

Definition 2. *([13]) Let v, b and k be three positive integers. A $(v,b,k,1)$-packing is a set system $(\mathcal{X},\mathcal{B})$ where $\mathcal{X}$ is a set of v elements and $\mathcal{B}$ is a set of b subsets of $\mathcal{X}$ called blocks that satisfy*

- *$|B| = k$ for any $B \in \mathcal{B}$;*
- *every pair of distinct elements of $\mathcal{X}$ occurs in at most one block of $\mathcal{B}$.*

A $(v,b,k,1)$-packing in which every pair of distinct elements of $\mathcal{X}$ occurs in exactly one block is called a balanced incomplete block design, or briefly $(v,b,k,1)$-BIBD.

Example 2. *(1) When $v = F = 6$, $K = 4$ and $Z = 3$, there exists a $(6,3,1)$ packing $(\mathcal{X},\mathcal{B})$ where*

$$\begin{aligned}\mathcal{X} &= \{0,1,2,3,4,5\}\\ \mathcal{B} &= \{\{3,4,5\},\{1,2,5\},\{0,2,4\},\{0,1,3\}\}.\end{aligned}$$

According to (7), we have

$$S(4,6,3) \geq \lceil (6-3)\times 4/6 \rceil + (6-3) - 1 = 4. \tag{8}$$

We can construct a $(4,6,3,4)$ PDA as follows.

$$\mathbf{P}_{6\times 4} = \begin{pmatrix} * & * & 0 & 1\\ * & 0 & * & 2\\ * & 1 & 2 & *\\ 0 & * & * & 3\\ 1 & * & 3 & *\\ 2 & 3 & * & * \end{pmatrix}$$

By (8), we can obtain that the $(4,6,3,4)$ PDA above is optimal given $F = 6$, $K = 4$ and $Z = 3$ which is exactly the MN PDA.

(2) When $F = 8$, $K = 6$ and $Z = 5$, there exists a $(8,3,1)$ packing $(\mathcal{X},\mathcal{B})$ where

$$\begin{aligned}\mathcal{X} =& \{0,1,2,3,4,5,6,7\}\\ \mathcal{B} =& \{\{0,1,3\},\{1,2,4\},\{2,3,5\},\{4,5,7\},\{5,6,0\},\{6,7,1\}\}.\end{aligned}$$

According to (7), we have

$$S(6,8,5) \geq \lceil (8-5)\times 6/8 \rceil + (8-5) - 1 = 5. \tag{9}$$

We can construct a $(6,8,5,5)$ PDA as follows:

$$\mathbf{P}_{8\times 6} = \begin{pmatrix} 0 & * & * & * & 3 & *\\ 1 & 3 & * & * & * & 4\\ * & 0 & 1 & * & * & *\\ 2 & * & 3 & * & * & *\\ * & 2 & * & 1 & * & *\\ * & * & 4 & 0 & 2 & *\\ * & * & * & * & 1 & 0\\ * & * & * & 3 & * & 2 \end{pmatrix}.$$

By (9), we can obtain that the $(6,8,5,5)$ PDA above is optimal given $F = 8$, $K = 6$ and $Z = 5$.

By the fact that for any two subsets A, B of $[F]$, $\overline{\overline{A} \bigcup \overline{B}} = A \bigcap B$ holds, Theorem 2 can also be written in the following way.

Corollary 1. *For any positive integers F, K and Z with $Z \leq F$, then the following inequality always holds.*

$$S(K,F,Z) \geq FK - \max_{\mathcal{A}_k\in\Omega, k\in[K]} \min\left\{ \sum_{h=0}^{K-1} |\bigcup_{j=0}^{h} \overline{\mathcal{A}}_{i_j}| \;\middle|\; (i_0,\ldots,i_{K-1}) \in \mathcal{I}\right\}. \tag{10}$$

IV. Conclusion

In this paper we proposed a lower bound on the value of S for any K, F and Z in a PDA. Then we showed that the value of S of the Group conjugate MN PDA in Lemma 3 and some PDAs generated by packings achieve this lower bound. However this lower bound is not always tight for any parameters K, F and Z. So it is meaningful to further improve this lower bound.

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