A Stochastic Hybrid Systems Approach to the Joint Distribution of Ages of Information in Networks

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Abstract—**We study a general setting of status updating systems in which a set of source nodes provide status updates about some physical process(es) to a set of monitors. The freshness of information available at each monitor is quantified in terms of the Age of Information (AoI), and the vector of AoI processes at the monitors (or equivalently the *age vector*) models the continuous state of the system. While the *marginal* distributional properties of each AoI process have been studied for a variety of settings using the stochastic hybrid system (SHS) approach, we lack a counterpart of this approach to systematically study their *joint* distributional properties. Developing such a framework is the main contribution of this paper. In particular, we model the discrete state of the system as a finite-state continuous-time Markov chain (MC), and describe the coupled evolution of the continuous and discrete states of the system by a piecewise linear SHS with linear reset maps. We start our analysis by deriving first-order linear differential equations for the temporal evolution of both the joint moments and the joint moment generating function (MGF) of all possible pairwise combinations formed by the age vector components. We then derive conditions under which the derived differential equations are asymptotically stable. Finally, we apply our framework to characterize the stationary joint MGF in a multi-source updating system under several queueing disciplines including non-preemptive and source-agnostic/source-aware preemptive in service queueing disciplines.**

Index Terms—**Age of information, queueing systems, communication networks, stochastic hybrid systems.**

I. Introduction

The explosive growth in the deployment of Internet of Things (IoT) is playing a pivotal role in enabling many critical real-time status updating systems that fundamentally rely on the timely delivery of status updates [1]. The authors of [2] introduced the concept of AoI which provides a rigorous way of quantifying the freshness of information at a *destination node* as a result of receiving status updates over time from a *transmitter node*. In particular, for a single-source queueing-theoretic model in which status updates are generated randomly at a transmitter with a single source of information, the AoI at the destination was defined in [2] as the following random process: $x(t) = t - u(t)$, where $u(t)$ is the generation time instant of the latest status update received at the destination by time t.

Following [2], the average value of AoI or peak AoI (a related metric based on the peak values of AoI over time) has been extensively analyzed in single-source systems under several queueing disciplines [3]–[5]. Meanwhile, the characterization of the average AoI in multi-source systems (where the transmitter has multiple sources of information) is quite challenging, and hence the prior work in this direction is relatively sparse [6]–[9]. Further, a handful of recent works have aimed to characterize the stationary distribution (or some distributional properties) of AoI/peak AoI in single-source [10]–[13] or multi-source [14] systems. Note that the analyses of the above works studying multi-source system settings (i.e., there are multiple AoI or age processes in the system) have been limited to the characterization of the marginal distributional properties of each source's AoI process.

The analyses of the above works were mainly based on identifying the properties of AoI sample functions and applying geometric arguments, which often involve tedious calculations of joint moments. Motivated by this, the authors of [15] and [16] have developed a SHS-based framework (building on [17]) for characterizing the marginal distributional properties of each AoI process in a network with multiple AoI processes. The results of [15] and [16] have then been applied to characterize the marginal distributional properties of AoI under a variety of queueing disciplines [18]–[22]. However, a systematic approach to the joint analysis of AoI processes is an open problem. In this paper, we develop an SHS-based general framework to facilitate the analysis of the joint distributional properties of the AoI processes through the characterization of their joint stationary moments and MGFs. Therefore, this paper is a joint distributional counterpart of [16]. It is instructive to note that a very recent prior work [23] has also analyzed the joint distributional properties of AoI processes in a particular bufferless multi-source single-server system setting using tools from Palm calculus. On the other hand, our framework is applicable to any generic queuing discipline. In fact, we will recover a key result of [23] as a special case of our analysis in Section IV.

Contributions. This paper presents a novel SHS-based framework for enabling the characterization of the stationary joint moments and joint MGFs of different AoI processes in networks. We first use the SHS framework to derive first-order linear differential equations for the temporal evolution of both the joint moments and the joint MGFs. We then demonstrate that the existence of the stationary joint first moment guarantees the existence of the stationary joint higher order moments and MGF. Afterwards, we apply our framework to derive closed-form expressions for the correlation coefficient of two AoI processes in a two-source updating system under both non-preemptive and preemptive in service queueing disciplines. Our analytical findings reveal that while the two AoI processes are negatively correlated under preemptive in service queueing disciplines for any choice of values of the system parameters, there exists a threshold value of server utilization in the non-preemptive queueing discipline above which the

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two age processes are positively correlated.

Notations. A vector $\mathbf{x} = [x_1 \ \cdots \ x_n]$ is a $1 \times n$ row vector with $[\mathbf{x}]_j = x_j$ denoting its j-th element. A matrix $\mathbf{X}$ has i,j-th element $[\mathbf{X}]_{i,j}$ and j-th column $[\mathbf{X}]_j$. The vectors $\mathbf{0}_n$ and $\mathbf{1}_n$ are the $1 \times n$ row vectors containing all zeros and ones, respectively, and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Whenever subscript n is dropped, the dimensions of $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{I}$ will be clear from the context. For a process $\mathbf{x}(t)$ or $\mathbf{X}(t)$, $\dot{\mathbf{x}}(t)$ or $\dot{\mathbf{X}}(t)$ denotes the derivative $\mathrm{d}\mathbf{x}(t)/\mathrm{d}t$ or $\mathrm{d}\mathbf{X}(t)/\mathrm{d}t$. For a scalar function $f(\cdot)$ and a vector $\mathbf{x}$, $f(\mathbf{x}) = [f(x_1) \ \cdots \ f(x_n)]$. For integers $m \le n, m : n = \{m, m+1, \cdots, n\}$. The Kronecker delta function $\delta_{i,j}$ equals 1 if $i = j$ and 0 otherwise. The vector $\mathbf{e}_i$ denotes the i-th Cartesian unit vector satisfying $[\mathbf{e}_i]_j = \delta_{i,j}$.

II. System Model and Problem Statement

A. Network Model

We consider a general setting of status updating systems where a set of source nodes provide status updates about some physical process(es) to a set of monitors. The freshness of information available at each monitor is quantified in terms of AoI. The AoI processes or equivalently the age processes in the system are modeled using the row vector $\mathbf{x}(t) = [x_1(t) \ \cdots \ x_n(t)]$, which is also referred to as the continuous state of the system. Further, the discrete state of the system is modeled using a finite-state continuous-time MC $q(t) \in \mathcal{Q} = \{0, \cdots, q_{max}\}$, where $\mathcal{Q}$ is the discrete state space. In the graphical representation of $q(t)$, each state $q \in \mathcal{Q}$ is a node and each transition l is a directed edge (q_l, q'_l) with fixed transition rate $\lambda^{(l)}(q(t)) = \lambda^{(l)}\delta_{q_l,q(t)}$, where the Kronecker delta function $\delta_{q_l,q(t)}$ ensures that transition l occurs only in state q_l. We denote the set of all transitions by $\mathcal{L}$, and the sets of incoming and outgoing transitions for state $\bar{q}$ by $\mathcal{L}'_{\bar{q}} = \{l \in \mathcal{L} : q'_l = \bar{q}\}$ and $\mathcal{L}_{\bar{q}} = \{l \in \mathcal{L} : q_l = \bar{q}\}$.

B. An SHS Formulation and Problem Statement

The coupled evolution of the continuous state $\mathbf{x}(t)$ and the discrete state $q(t)$ is modeled using a piecewise linear stochastic hybrid system with linear reset maps [16]. In particular, when a transition l occurs in the MC $q(t)$, the continuous state $\mathbf{x}$ is reset to $\mathbf{x}'$ according to a reset map matrix $\mathbf{A}_l$ as $\mathbf{x}' = \mathbf{x}\mathbf{A}_l$. Further, as long as the state $q(t)$ is unchanged, each component in the age vector $\mathbf{x}(t)$ grows at a unit rate with time (which yields piecewise linear age processes over time), i.e., $\dot{\mathbf{x}}(t) \triangleq \frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{1}$. To capture the temporal evolution of the age processes, it is sufficient to assume that $\mathbf{A}_l$ is a binary matrix with no more than a single 1 in a column. Since column $[\mathbf{A}_l]_j$ determines the value that will be assigned to x'_j, we have two different cases given the assumed structure of $\mathbf{A}_l$. In the first case, $[\mathbf{A}_l]_j = \mathbf{0}$ and so $x'_j = 0$, whereas the second case corresponds to $[\mathbf{A}_l]_j = \mathbf{e}_i^{\mathrm{T}}$ where x'_j is reset to x_i. Different from ordinary continuous-time Markov Chains, an inherent feature of SHS is the possibility of having self-transitions in the MC $q(t)$ modeling the system discrete state. In particular, although a self-transition keeps $q(t)$ unchanged, it causes a change in the continuous state $\mathbf{x}(t)$. Further, there may be multiple transitions between any two states in $\mathcal{Q}$ such that their associated reset map matrices are different.

For the above SHS formulation, our prime objective in this paper is to develop a framework that allows understanding/analyzing the joint distributional properties of the components in the age vector $\mathbf{x}(t)$. In particular, we aim at characterizing the stationary joint moments and joint MGFs which are of the following forms: $\lim_{t\to\infty} \mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)]$ and $\lim_{t\to\infty} \mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}]$, $\forall j, k \in 1 : n$ and $m_1, m_2 \ge 1$. Clearly, the characterization of such joint moments and joint MGFs allows one to derive the correlation coefficient between all possible pairwise combinations of the age vector components. Given the generality of the system setting considered in this paper, the importance of our framework lies in the fact that it is applicable to the joint analysis of AoIs in a broad range of status updating system setups under arbitrary queueing disciplines.

III. Joint Analysis of Age Processes in Networks

A. Differential Equations for the Temporal Evolution of the Joint Moments and Joint MGFs

In order to characterize the temporal evolution of the joint moments and joint MGFs, $\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)]$ and $\mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}]$, it is useful to define the following quantities that express different forms of correlation between $q(t)$ and the age processes in $\mathbf{x}(t)$:

$$v_{\bar{q},j}^{(m)}(t) = \mathbb{E}[x_j^m(t)\delta_{\bar{q},q(t)}], \quad (1)$$

$$v_{\bar{q},j}^{(s)}(t) = \mathbb{E}[e^{s x_j(t)}\delta_{\bar{q},q(t)}], \quad (2)$$

$$v_{\bar{q},jk}^{(m_1,m_2)}(t) = \mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)\delta_{\bar{q},q(t)}], \quad (3)$$

$$v_{\bar{q},jk}^{(s_1,s_2)}(t) = \mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}\delta_{\bar{q},q(t)}], \quad (4)$$

for all states $\bar{q} \in \mathcal{Q}$, $j,k \in 1 : n$, $m \ge 0$, and $m_1, m_2 \ge 1$. To see this, note that $\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)]$ and $\mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}]$ can respectively be expressed as

$$\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)] = \sum_{\bar{q}\in\mathcal{Q}} \underbrace{\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)\delta_{\bar{q},q(t)}]}_{v_{\bar{q},jk}^{(m_1,m_2)}(t)}, \quad (5)$$

$$\mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}] = \sum_{\bar{q}\in\mathcal{Q}} \underbrace{\mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}\delta_{\bar{q},q(t)}]}_{v_{\bar{q},jk}^{(s_1,s_2)}(t)}. \quad (6)$$

Thus, according to (5) and (6), characterizing the temporal evolution of $v_{\bar{q},jk}^{(m_1,m_2)}(t)$ and $v_{\bar{q},jk}^{(s_1,s_2)}(t)$ directly characterizes the temporal evolution of $\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)]$ and $\mathbb{E}[e^{s_1 x_j(t) + s_2 x_k(t)}]$, respectively. Some key notes about the notations in (1)-(4) are provided next. First, $v_{\bar{q},j}^{(1)}$ may generally refer to $v_{\bar{q},j}^{(m)}|_{m=1}$ or $v_{\bar{q},j}^{(s)}|_{s=1}$. To eliminate this conflict, the convention that $v_{\bar{q},j}^{(i)}$, for any integer $i \ge 1$, refers to $v_{\bar{q},j}^{(m)}$ at $m = i$ is maintained here. The previous argument also applies to $v_{\bar{q},jk}^{(m_1,m_2)}$ and $v_{\bar{q},jk}^{(s_1,s_2)}$. Further, note that $v_{\bar{q},jk}^{(s_1,0)} = v_{\bar{q},j}^{(s)}|_{s=s_1}$ and $v_{\bar{q},jk}^{(0,s_2)} = v_{\bar{q},k}^{(s)}|_{s=s_2}$. Finally, we have $v_{\bar{q},j}^{(m)}(t)|_{m=0} = v_{\bar{q},j}^{(s)}(t)|_{s=0} = \mathbb{E}[\delta_{\bar{q},q(t)}] = \mathbb{P}[q(t) = \bar{q}]$, i.e., $v_{\bar{q},j}^{(0)}(t)$ refers to the probability that $q(t)$ is equal to $\bar{q}$ regardless of the value of j. It will also be useful in our subsequent analysis and exposition to define following vectors/matrices containing the scalars in (1)-(4): $[\mathbf{v}_{\bar{q}}^{(0)}(t)]_j = v_{\bar{q},j}^{(0)}(t)$, $[\mathbf{v}_{\bar{q}}^{(m)}(t)]_j = v_{\bar{q},j}^{(m)}(t)$, $[\mathbf{v}_{\bar{q}}^{(s)}(t)]_j = v_{\bar{q},j}^{(s)}(t)$, $[\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)]_{j,k} = v_{\bar{q},jk}^{(m_1,m_2)}(t)$ and $[\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)]_{j,k} = v_{\bar{q},jk}^{(s_1,s_2)}(t), \forall j,k \in 1:n, \bar{q} \in \mathcal{Q}$. The following Lemma shows that $\{v_{\bar{q},jk}^{(m_1,m_2)}(t)\}$ and $\{v_{\bar{q},jk}^{(s_1,s_2)}(t)\}$ obey a system of first-order ordinary differential equations.

Lemma 1. *For state $\bar{q} \in \mathcal{Q}$ in the piecewise linear stochastic hybrid system with linear reset maps under consideration,*

$$\dot{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}(t) = m_1\mathbf{V}_{\bar{q}}^{(m_1-1,m_2)}(t) + m_2\mathbf{V}_{\bar{q}}^{(m_1,m_2-1)}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)}\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{q_l}^{(m_1,m_2)}(t)\mathbf{A}_l - \mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (7)$$

$$\dot{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)}(t) = (s_1+s_2)\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t) + \mathbf{C}_{\bar{q}}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)}\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{q_l}^{(s_1,s_2)}(t)\mathbf{A}_l - \mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (8)$$

where $[\mathbf{C}_{\bar{q}}(t)]_{j,k} = c_{\bar{q},jk}(t)$ is defined as

$$c_{\bar{q},jk}(t) = \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\Big[\mathbf{1}\big([\mathbf{x}\mathbf{A}_l]_j \neq 0 \;\&\; [\mathbf{x}\mathbf{A}_l]_k = 0\big) \times [\mathbf{v}_{q_l}^{(s_1)}(t)\mathbf{A}_l]_j + \mathbf{1}\big([\mathbf{x}\mathbf{A}_l]_j = 0 \;\&\; [\mathbf{x}\mathbf{A}_l]_k \neq 0\big)[\mathbf{v}_{q_l}^{(s_2)}(t)\mathbf{A}_l]_k + \mathbf{1}\big([\mathbf{x}\mathbf{A}_l]_j = 0 \;\&\; [\mathbf{x}\mathbf{A}_l]_k = 0\big)[\mathbf{v}_{q_l}^{(0)}]_j\Big], \quad (9)$$

where $\&$ is the logical AND operator and $\mathbf{1}(\cdot)$ is the indicator function.

Proof: See Appendix A. ■

In order to clearly see that Lemma 1 characterizes the trajectories of $\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)$ and $\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)$ over time, it is useful to first state the following differential equations of [16, Lemma 1] characterizing the temporal evolution of the marginal m-th moments and marginal MGFs:

$$\dot{\mathbf{v}}_{\bar{q}}^{(0)}(t) = \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\mathbf{v}_{q_l}^{(0)}(t) - \mathbf{v}_{\bar{q}}^{(0)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (10)$$

$$\dot{\mathbf{v}}_{\bar{q}}^{(m)}(t) = m\mathbf{v}_{\bar{q}}^{(m-1)}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\mathbf{v}_{q_l}^{(m)}(t)\mathbf{A}_l - \mathbf{v}_{\bar{q}}^{(m)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \forall m \geq 1, \quad (11)$$

$$\dot{\mathbf{v}}_{\bar{q}}^{(s)}(t) = s\mathbf{v}_{\bar{q}}^{(s)}(t) + \sum_{l\in\mathcal{L}'_{\bar{q}}}\lambda^{(l)}\big[\mathbf{v}_{q_l}^{(s)}(t)\mathbf{A}_l + \mathbf{v}_{q_l}^{(0)}(t)\hat{\mathbf{A}}_l\big] - \mathbf{v}_{\bar{q}}^{(s)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (12)$$

$$[\hat{\mathbf{A}}]_{i,j} = \begin{cases} 1 & i=j, [\mathbf{A}_l]_j = \mathbf{0}^{\mathrm{T}}, \\ 0 & \text{otherwise}. \end{cases} \quad (13)$$

We are now ready to elaborate on the use of Lemma 1 to obtain the trajectories of $\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)$ and $\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)$. For a given initial condition at $t=0$, we observe from (7) that in order to compute $\mathbf{V}_{\bar{q}}^{(1,1)}(t)$, we need to first compute $\mathbf{V}_{\bar{q}}^{(0,1)}(t)$ and $\mathbf{V}_{\bar{q}}^{(1,0)}(t)$ using (11). This can be done after computing $\mathbf{v}_{\bar{q}}^{(0)}(t)$ from (10). Afterwards, $\mathbf{V}_{\bar{q}}^{(2,1)}(t)$ can be computed from $\mathbf{V}_{\bar{q}}^{(1,1)}(t)$ and $\mathbf{V}_{\bar{q}}^{(2,0)}(t)$, where $\mathbf{V}_{\bar{q}}^{(2,0)}(t)$ can be evaluated using (11). The process can be repeated to compute $\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)$ for the desired $m_1, m_2 \geq 2$ using $\mathbf{V}_{\bar{q}}^{(m_1-1,m_2)}(t)$ and $\mathbf{V}_{\bar{q}}^{(m_1,m_2-1)}(t)$ evaluated in previous steps. Further, by inspecting the structure of $\mathbf{C}_{\bar{q}}(t)$ from (9), we note that $\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)$ can be computed from $\mathbf{v}_{\bar{q}}^{(s)}(t)$ and $\mathbf{v}_{\bar{q}}^{(0)}(t)$, where $\mathbf{v}_{\bar{q}}^{(s)}(t)$ can be evaluated from $\mathbf{v}_{\bar{q}}^{(0)}(t)$ using (12).

B. Stationary Joint Moments and Joint MGFs

While Lemma 1 holds for any collection of reset map matrices $\{\mathbf{A}_l\}_{l\in\mathcal{L}}$, the set of differential equations in Lemma 1 can be unstable for some choices of $\{\mathbf{A}_l\}_{l\in\mathcal{L}}$. Thus, it is necessary to investigate the conditions under which the differential equations in Lemma 1 are stable. While there are several notions of stability including Lyapunov, Lagrange, and exponential stability, we are interested here in the asymptotic stability under which $\dot{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}(t)$ and $\dot{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)}(t)$ respectively converge to the limits $\bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}$ and $\bar{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)}$ as $t\to\infty$. The limiting values can then be evaluated as the solution of the equations resulting from setting the derivatives in Lemma 1 to zero. To clearly see why we are concerned about the asymptotic stability in this paper, recall that our prime objective is to characterize the stationary joint moments and joint MGFs: $\lim_{t\to\infty}\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)]$ and $\lim_{t\to\infty}\mathbb{E}[e^{s_1x_j(t)+s_2x_k(t)}]$, $\forall j,k \in 1:n$. Under the asymptotic stability, these quantities can simply be evaluated from (5) and (6) as

$$\lim_{t\to\infty}\mathbb{E}[x_j^{m_1}(t)x_k^{m_2}(t)] = \sum_{\bar{q}\in\mathcal{Q}}\lim_{t\to\infty}v_{\bar{q},jk}^{(m_1,m_2)}(t) = \sum_{\bar{q}\in\mathcal{Q}}\bar{v}_{\bar{q},jk}^{(m_1,m_2)}, \quad (14)$$

$$\lim_{t\to\infty}\mathbb{E}[e^{s_1x_j(t)+s_2x_k(t)}] = \sum_{\bar{q}\in\mathcal{Q}}\lim_{t\to\infty}v_{\bar{q},jk}^{(s_1,s_2)}(t) = \sum_{\bar{q}\in\mathcal{Q}}\bar{v}_{\bar{q},jk}^{(s_1,s_2)}. \quad (15)$$

We now proceed to characterizing the conditions under which the differential equations in Lemma 1 are asymptotically stable. Let us first recall the asymptotic stability theorem for linear systems. The linear system

$$\dot{\mathbf{v}}(t) = \mathbf{v}(t)\mathbf{P},\ \mathbf{v}(0) = \mathbf{v}_0 \quad (16)$$

is asymptotically stable if and only if the eigenvalues of $\mathbf{P}$ have strictly negative real parts. According to (16), it is always useful to write the differential equations at hand in a vector form to test the asymptotic stability. This was also done in [16] to characterize the conditions under which the differential equations describing the temporal evolution of the marginal m-th moments and marginal MGFs (given by (11) and (12)) are asymptotically stable. For all $\bar{q}\in\mathcal{Q}$, let $\bar{\mathbf{v}}_{\bar{q}}^{(0)}$, $\bar{\mathbf{v}}_{\bar{q}}^{(m)}$ and $\bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}$ denote the limiting values of $\mathbf{v}_{\bar{q}}^{(0)}(t)$, $\mathbf{v}_{\bar{q}}^{(m)}(t)$ and $\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)$, respectively, when $t\to\infty$. Clearly, $\bar{\mathbf{v}}_{\bar{q}}^{(0)}$, $\bar{\mathbf{v}}_{\bar{q}}^{(m)}$ and $\bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}$ are the fixed points of (10), (11) and (7), respectively, which can be obtained after setting the derivatives to zero. The next theorem characterizes the conditions for asymptotic stability of the equations in Lemma 1.

Theorem 1. *If the MC $q(t)$ is ergodic with stationary distribution $\bar{\mathbf{v}}_{\bar{q}}^{(0)} > 0$, and there exist positive fixed points $\bar{\mathbf{v}}_{\bar{q}}^{(1)}$ and $\bar{\mathbf{V}}_{\bar{q}}^{(1,1)}$ of (11) and (7), respectively, then:*

- *(i) For all $\bar{q} \in \mathcal{Q}$, $\mathbf{V}_{\bar{q}}^{(m_1,m_2)}(t)$ converges to $\bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)}$ satisfying*

$$\bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2)} \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)} = m_1 \bar{\mathbf{V}}_{\bar{q}}^{(m_1-1,m_2)} + m_2 \bar{\mathbf{V}}_{\bar{q}}^{(m_1,m_2-1)} + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \mathbf{A}_l^{\mathrm{T}} \bar{\mathbf{V}}_{q_l}^{(m_1,m_2)} \mathbf{A}_l. \tag{17}$$

- *(ii) There exists $s_0 > 0$ such that for all $(s_1, s_2) \in \mathcal{S} = \{(s_1, s_2) : s_1 + s_2 < s_0\}$ and $\bar{q} \in \mathcal{Q}$, $\mathbf{V}_{\bar{q}}^{(s_1,s_2)}(t)$ and $\mathbf{C}_{\bar{q}}(t)$ respectively converge to $\bar{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)}$ and $\bar{\mathbf{C}}_{\bar{q}}$ satisfying*

$$\bar{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)} \sum_{l\in\mathcal{L}_{\bar{q}}} \lambda^{(l)} = (s_1 + s_2) \bar{\mathbf{V}}_{\bar{q}}^{(s_1,s_2)} + \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \mathbf{A}_l^{\mathrm{T}} \bar{\mathbf{V}}_{q_l}^{(s_1,s_2)} \mathbf{A}_l + \bar{\mathbf{C}}_{\bar{q}}. \tag{18}$$

Proof: See Appendix B. ■

Theorem 1 is a generalized version of [16, Theorem 1] which was focused on the characterization of the stationary marginal moments and MGFs, i.e., the fixed points of (11) and (12). An interesting analogy between Theorem 1 and [16, Theorem 1] is that the existence of the stationary marginal/joint first moment guarantees the existence of the stationary marginal/joint higher order moments and MGF. It is worth emphasizing that the generality of Theorem 1 lies in its ability of allowing the investigation of the stationary joint moments and MGFs under any arbitrary queueing discipline. This opens the door for the use of Theorem 1 to study the joint analysis of age processes in networks for different queueing disciplines/status updating system settings in the literature, which have been only analyzed in terms of the marginal moments and MGFs until now.

IV. Analysis of the Stationary Joint MGF in Multi-source Updating Systems

In this section, we use Theorem 1 to analyze the stationary joint MGF of the age processes in a multi-source status updating system, where a transmitter monitors two physical processes, and sends its measurements to a destination in the form of status updates. As shown in Fig. 1, the transmitter consists of two sources and a single server; each source generates status updates about one physical process, and the server delivers the status updates generated from the sources to the destination. Status updates generated by the i-th source are assumed to follow a Poisson process with rate λ_i. Further, the time needed by the server to send a status update is assumed to be a rate μ exponential random variable. Let $\rho = \frac{\lambda}{\mu}$ denote the server utilization factor, where $\lambda = \lambda_1 + \lambda_2$. Further, we have $\rho_i = \frac{\lambda_i}{\mu}$, $\lambda_{-i} = \sum_{j=1,\, j\neq i}^{2} \lambda_j$, and $\rho_{-i} = \frac{\lambda_{-i}}{\mu}$. We derive the joint MGF of the two age processes (associated with the two observed physical processes) at the destination under three different queueing disciplines for managing status update arrivals at the transmitter, which are described next.

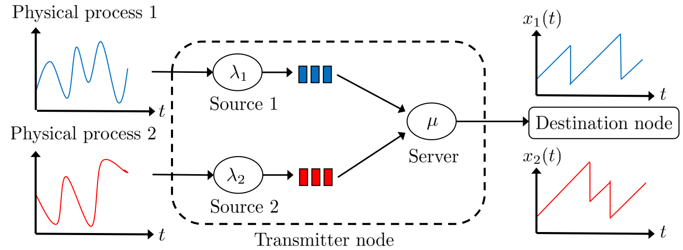

Fig. 1. An illustration of a two-source status updating system.

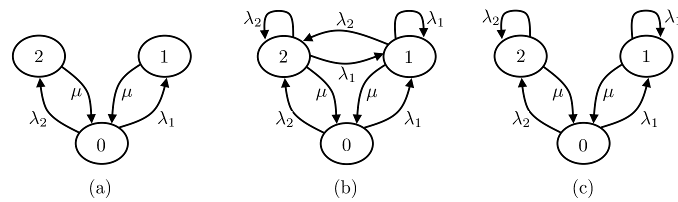

Fig. 2. Markov chains modeling the system discrete state $q(t)$ under different queueing disciplines: (a) LCFS-NP, (b) LCFS-PS, and (c) LCFS-SA.

Last-come-first-served with no preemption (LCFS-NP): Under this queueing discipline, a new arriving status update at the transmitter (from any of the sources) enters service upon its arrival if the server is idle (i.e., there is no a status update in service); otherwise, the new arriving status update is discarded.

Last-come-first-served with source-agnostic preemption in service (LCFS-PS): When the server is idle, the management of a new arriving status update under this queueing discipline is similar to the LCFS-NP one. However, when the server is busy, a new arriving status update replaces the current update being served (regardless of the index of its generating source) and the old update in service is discarded.

Last-come-first-served with source-aware preemption in service (LCFS-SA): This queueing discipline is similar to the LCFS-PS one with the only difference that a new arriving status update preempts the update in service only if the two updates (the new arriving update and the one in service) are generated from the same source.

Using the notations of the SHS framework, the continuous state $\mathbf{x}(t)$ in each queueing discipline is given by $\mathbf{x}(t) = [x_1(t)\ x_2(t)\ x_3(t)]$, where $x_i(t), i \in \{1, 2\}$, represents the value of the source i's AoI at the destination node, and $x_3(t)$ is the age of the status update in service. Further, the discrete state space in each queueing discipline is given by $\mathcal{Q} = \{0, 1, 2\}$, where $q(t) = 0$ indicates that the system is empty and hence the server is idle, and $q(t) = i, i \in \{1, 2\}$, indicates that the server is serving a status update generated from the i-th source. Further, the continuous-time MC modeling the system discrete state $q(t) \in \mathcal{Q}$ under each of the queueing disciplines is depicted in Fig. 2.

A. LCFS-NP Queueing Discipline

Table I presents the set of different transitions $\mathcal{L}$ and their impact on the values of both $q(t)$ and $\mathbf{x}(t)$. Before proceeding into evaluating $\bar{v}_{\bar{q},12}^{(s_1,s_2)}$, $\forall \bar{q} \in \mathcal{Q}$, satisfying (18), we first describe the set of transitions as follows:

TABLE I
TRANSITIONS OF THE LCFS-NP QUEUEING DISCIPLINE IN FIG. 2A.

l	$q_l \rightarrow q'_l$	$\lambda^{(l)}$	$\mathbf{x}\mathbf{A}_l$	$\mathbf{A}_l$
1	$0 \rightarrow 1$	λ_1	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
2	$0 \rightarrow 2$	λ_2	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
3	$1 \rightarrow 0$	μ	$[x_3\ x_2\ 0]$	$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}$
4	$2 \rightarrow 0$	μ	$[x_1\ x_3\ 0]$	$\begin{bmatrix}1&0&0\\0&0&0\\0&1&0\end{bmatrix}$

$l = i, i \in \{1, 2\}$: This transition occurs when there is a new arriving status update of source i at the transmitter node. Since the age of this new arriving status update at the transmitter is 0 and it does not have any impact on the AoI processes of the two sources at the destination, we note that the updated age vector $\mathbf{x}\mathbf{A}_i$ is set to be $[x_1\ x_2\ 0]$.

$l \in \{3, 4\}$: This transition occurs when the status update in service is delivered to the destination. When the status update received at the destination belongs to source i, the AoI of source i is reset to its age and the AoI of the other source does not change. For instance when $l = 3$, a status update of source 1 is received at the destination. Hence, we note that the source 1's AoI is reset to the age of the received update ($[\mathbf{x}\mathbf{A}_3]_1 = x_3$) whereas the source 2's AoI does not change ($[\mathbf{x}\mathbf{A}_3]_2 = x_2$). In addition, since the system becomes empty after the occurrence of this transition, the third component of the age vector $\mathbf{x}(t)$ becomes irrelevant. Following the convention [15], we set the value corresponding to such irrelevant components in the updated age value to 0, and thus we observe that $[\mathbf{x}\mathbf{A}_3]_3 = 0$.

Using Table I, we are now ready to derive $\{\bar{v}_{\bar{q},12}^{(s_1,s_2)}\}_{\bar{q}\in\mathcal{Q}}$ satisfying (18), form which the stationary joint MGF is characterized in the following theorem.

Theorem 2. *Under the LCFS-NP queueing discipline, the stationary joint MGF is given by*

$$\overset{\mathrm{NP}}{M}(\bar{s}_1, \bar{s}_2) = \frac{\rho_1\rho_2\big[1 + \rho - (\bar{s}_1 + \bar{s}_2)\big]}{(1+\rho)\big[\rho - (\bar{s}_1 + \bar{s}_2)\big]\big[1 - (\bar{s}_1 + \bar{s}_2)\big]^2} \times \sum_{i=1}^{2} \frac{1}{(1-\bar{s}_i)(\rho - \bar{s}_i) - \rho_{-i}}, \tag{19}$$

where $\bar{s}_i = \frac{s_i}{\mu}, i \in \{1, 2\}$.

Proof: See Appendix C. ■

Remark 1. *Note that the marginal MGF of source i's AoI under the LCFS-NP queueing discipline can be obtained from the joint MGF in (19) by setting $\bar{s}_j$ ($j \neq i$) to zero. This argument applies to any queueing discipline under consideration including the ones presented next.*

Corollary 1. *Under the LCFS-NP queueing discipline, the correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ is given by*

$$\overset{\mathrm{NP}}{\mathrm{Cor}} = \frac{\rho_1\rho_2\big[\rho^3 - 2(2\rho + 1)\big]}{\rho\prod_{i=1}^{2}\sqrt{(1+\rho)^2[\rho^2 + 2\rho_{-i} + 1] + \rho_i^2\rho(\rho+2)}}. \tag{20}$$

TABLE II
TRANSITIONS OF THE LCFS-PS QUEUEING DISCIPLINE IN FIG. 2B.

l	$q_l \rightarrow q'_l$	$\lambda^{(l)}$	$\mathbf{x}\mathbf{A}_l$	$\mathbf{A}_l$
5	$1 \rightarrow 2$	λ_2	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
6	$2 \rightarrow 1$	λ_1	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
7	$1 \rightarrow 1$	λ_1	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$
8	$2 \rightarrow 2$	λ_2	$[x_1\ x_2\ 0]$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$

Proof: The correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ can be evaluated as

$$\overset{\mathrm{NP}}{\mathrm{Cor}} = \frac{\mathbb{E}[x_1x_2] - \mathbb{E}[x_1]\mathbb{E}[x_2]}{\sqrt{\mathbb{E}[x_1^2] - (\mathbb{E}[x_1])^2}\sqrt{\mathbb{E}[x_2^2] - (\mathbb{E}[x_2])^2}}. \tag{21}$$

The expression in (20) then follows from the fact that the expression of $\overset{\mathrm{NP}}{M}(\bar{s}_1, \bar{s}_2)$ (derived in Theorem 2) can be used to compute the stationary moments in (21) as follows

$$\mathbb{E}[x_1^{m_1}x_2^{m_2}] = \frac{\partial^{m_1+m_2}\big[\overset{\mathrm{NP}}{M}(\bar{s}_1, \bar{s}_2)\big]}{\mu^{m_1+m_2}\partial\bar{s}_1^{m_1}\partial\bar{s}_2^{m_2}}\Big|_{\bar{s}_1=0,\bar{s}_2=0}, \forall m_1, m_2 \geq 1, \tag{22}$$

$$\mathbb{E}[x_1^{m_1}] = \frac{1}{\mu^{m_1}} \times \frac{\mathrm{d}^{m_1}\big[\overset{\mathrm{NP}}{M}(\bar{s}_1, 0)\big]}{\mathrm{d}\bar{s}_1^{m_1}}\Big|_{\bar{s}_1=0}, \forall m_1 \geq 1, \tag{23}$$

$$\mathbb{E}[x_2^{m_2}] = \frac{1}{\mu^{m_2}} \times \frac{\mathrm{d}^{m_2}\big[\overset{\mathrm{NP}}{M}(0, \bar{s}_2)\big]}{\mathrm{d}\bar{s}_2^{m_2}}\Big|_{\bar{s}_2=0}, \forall m_2 \geq 1. \tag{24}$$

■

B. LCFS-PS Queueing Discipline

The set of transitions in the LCFS-PS queueing discipline can be constructed using Tables I and II. The subset of transitions in Table II refers to the event of having a new arriving status update at the transmitter node while its server is serving another status update. According to the mechanism of the LCFS-PS queueing discipline, the status update that is currently being served will be discarded, and the new arrival will enter service upon its arrival. The stationary joint MGF for this case is provided in the next theorem.

Theorem 3. *Under the LCFS-PS queueing discipline, the stationary joint MGF is given by*

$$\overset{\mathrm{PS}}{M}(\bar{s}_1, \bar{s}_2) = \frac{\rho_1\rho_2}{\big[\rho - (\bar{s}_1 + \bar{s}_2)\big]\big[1 - (\bar{s}_1 + \bar{s}_2)\big]} \times \sum_{i=1}^{2} \frac{1}{(1-\bar{s}_i)(\rho - \bar{s}_i) - \rho_{-i}}. \tag{25}$$

Proof: The proof follows along similar lines to the proof of Theorem 2. In particular, Tables I and II will be used to derive $\bar{v}_{\bar{q},12}^{(s_1,s_2)}$ satisfying (18), form which the stationary joint MGF in (25) can be obtained. ■

Corollary 2. *Under the LCFS-PS queueing discipline, the correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ is given by*

$$\overset{\mathrm{PS}}{\mathrm{Cor}} = \frac{-2\rho_1\rho_2}{\rho\sqrt{(\rho^2 + 2\rho_1 + 1)(\rho^2 + 2\rho_2 + 1)}}. \tag{26}$$

Remark 2. *Note that the expression in (26) is identical to the correlation coefficient expression derived in [23, Theorem 2] using tools from Palm calculus (for a two-source system setting under the LCFS-PS queueing discipline).*

C. LCFS-SA Queueing Discipline

The set of transitions in this queueing discipline can be constructed using Tables I and II as $\mathcal{L} = \{1, 2, 3, 4, 7, 8\}$. Note that $l = 5$ and $l = 6$ were excluded from $\mathcal{L}$ since the LCFS-SA queueing discipline only allows preemption between the status updates generated from the same source. In the next theorem, we provide the stationary joint MGF.

Theorem 4. *Under the LCFS-SA queueing discipline, the stationary joint MGF is given by:* $\overset{\text{SA}}{M}(\bar{s}_1, \bar{s}_2) =$

$$\frac{\rho_1\rho_2\big[1+\rho-(\bar{s}_1+\bar{s}_2)\big]}{(1+\rho)\big[\rho-(\bar{s}_1+\bar{s}_2)\big]\big[1-(\bar{s}_1+\bar{s}_2)\big]}\sum_{i=1}^{2}\Bigg[\frac{(1+\rho_i)}{(1+\rho_i-\bar{s}_i)}\times \frac{(1+\rho_{-i}-\bar{s}_i)}{\big[1+\rho_{-i}-(\bar{s}_1+\bar{s}_2)\big]\big[(1-\bar{s}_i)(\rho-\bar{s}_i)-\rho_{-i}\big]}\Bigg]. \tag{27}$$

Proof: The proof follows along similar lines to the proof of Theorem 2. The detailed proof has been omitted for the sake of brevity. ■

Corollary 3. *Under the LCFS-SA queueing discipline, the correlation coefficient of the two AoI processes $x_1(t)$ and $x_2(t)$ is given by*

$$\overset{\text{SA}}{\text{Cor}} = \frac{-\rho_1\rho_2 g(\rho_1,\rho_2)}{\rho(1+\rho_1)(1+\rho_2)\sqrt{f(\rho_1,\rho_2)f(\rho_2,\rho_1)}}, \tag{28}$$

where $g(\rho_1, \rho_2)$ and $f(y, z)$ are respectively given by:

$$g(\rho_1,\rho_2) = \rho_1^2\rho_2^2(\rho+2)(2\rho+1)+\rho_1\rho_2\rho(1+\rho)(3\rho+5) + 2(1+\rho)^4, \tag{29}$$

$$f(y,z) = z^3y + y^2z(2\rho^2+7\rho+4) + yz(\rho^2+6\rho+3) + y^2\rho^3(\rho+2) + y\rho(2\rho^3+6\rho^2+4\rho+1) + (1+\rho)^4. \tag{30}$$

Remark 3. *From Corollaries 1-3, we note that while the two age processes are negatively correlated under preemptive in service queueing disciplines (LCFS-PS and LCFS-SA) for any choice of values of the system parameters, there exists a threshold value $\rho_{\text{th}} \approx 2.2143$ of ρ in the non-preemptive queueing discipline (LCFS-NP) above which the two age processes are positively correlated. Further, we observe from Fig. 3 that the source-aware preemption in service slightly reduces the negative correlation of the two age processes compared to the source-agnostic one.*

V. Conclusion

This paper presented a novel SHS-based framework that allows the analysis of the joint distributional properties of AoI processes in networks through the characterization of the joint stationary moments and MGFs. An interesting insight drawn from our analysis is that the existence of the stationary joint first moment guarantees the existence of the stationary joint higher order moments and MGF. As an application of our framework, we obtained closed-from expressions of the stationary joint MGF in a two-source updating system under several queueing disciplines including non-preemptive and source-agnostic/source-aware preemptive in service queueing disciplines. Our derived expressions demonstrated that while the two AoI processes are negatively correlated under preemptive in service queueing disciplines for any choice of values of the system parameters, there exists a threshold value of server utilization in the non-preemptive queueing discipline above which the two age processes are positively correlated. Further, we numerically demonstrated that the source-aware preemption in service slightly reduces the negative correlation of the two age processes compared to the source-agnostic one.

The generality of our analytical framework stems from the fact that it allows one to understand the joint distributional properties of AoI processes in a broad range of system settings under any arbitrary queueing discipline. This, in turn, opens the door for the use of our framework in future work to investigate the joint stationary moments and MGFs of age processes for different queueing disciplines/status updating system settings in the literature, which have been only analyzed in terms of the marginal moments and MGFs until now.

Appendix

A. Proof of Lemma 1

To derive this result, we follow a similar approach to that in [16] and [17], where the idea is to define test functions $\{\psi(q, \mathbf{x})\}$ whose expected values $\{\mathbb{E}[\psi(q(t), \mathbf{x}(t))]\}$ are quantities of interest. Then, one can use the SHS framework to derive a system of differential equations for the temporal evolution of the expected values of the defined test functions. Since we are interested in the joint analysis of age processes in this paper, we define the following two classes of test functions

$$\psi_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x}) = x_j^{m_1}x_k^{m_2}\delta_{\bar{q},q}, \forall \bar{q}\in\mathcal{Q}, \text{ and } j,k\in 1:n, \tag{31}$$

$$\psi_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x}) = e^{s_1x_j+s_2x_k}\delta_{\bar{q},q}, \forall \bar{q}\in\mathcal{Q}, \text{ and } j,k\in 1:n. \tag{32}$$

Clearly, taking the expectation of the two classes of test functions in (31) and (32) gives $\{v_{\bar{q},jk}^{(m_1,m_2)}(t)\}$ and $\{v_{\bar{q},jk}^{(s_1,s_2)}(t)\}$, respectively. Now, we apply the SHS mapping $\psi \to L\psi$ (known as the extended generator) to every test function in (31) and (32). Since the test functions defined above are time-invariant, it follows from [17, Theorem 1] that the extended generator of the considered piecewise linear SHS with linear reset maps is given by

$$L\psi(q,\mathbf{x}) = \frac{\partial\psi(q,\mathbf{x})}{\partial\mathbf{x}}\mathbf{1}^{\mathrm{T}} + \underbrace{\sum_{l\in\mathcal{L}}\lambda^{(l)}(q)\big[\psi(q_l',\mathbf{x}\mathbf{A}_l)-\psi(q,\mathbf{x})\big]}_{\theta(q,\mathbf{x})}, \tag{33}$$

where the row vector $\partial\psi(q,\mathbf{x})/\partial\mathbf{x}$ denotes the gradient. Applying (33) to the test functions in (31) and (32), we have

$$L\psi_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x}) = \frac{\partial\psi_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x})}{\partial\mathbf{x}}\mathbf{1}^{\mathrm{T}} + \theta_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x}), \tag{34}$$

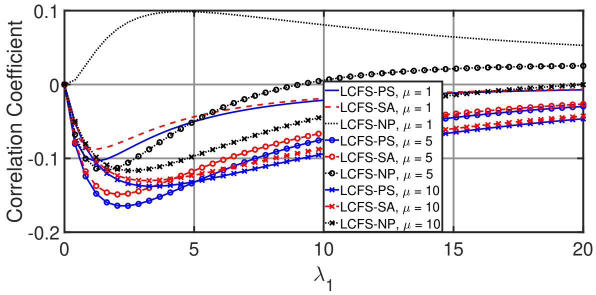

Fig. 3. Correlation coefficient of the two AoI processes as a function of λ_1 for a fixed $\lambda_2 = 2$ and different values of μ.

$$L\psi_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x}) = \frac{\partial \psi_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x})}{\partial \mathbf{x}}\mathbf{1}^{\mathrm{T}} + \theta_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x}), \quad (35)$$

$$\frac{\partial \psi_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x})}{\partial \mathbf{x}} = m_1\psi_{\bar{q},jk}^{(m_1-1,m_2)}\mathbf{e}_j + m_2\psi_{\bar{q},jk}^{(m_1,m_2-1)}\mathbf{e}_k, \quad (36)$$

$$\frac{\partial \psi_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x})}{\partial \mathbf{x}} = s_1\psi_{\bar{q},jk}^{(s_1,s_2)}\mathbf{e}_j + s_2\psi_{\bar{q},jk}^{(s_1,s_2)}\mathbf{e}_k. \quad (37)$$

Now, to obtain $\theta_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x})$ and $\theta_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x})$, note that

$$\psi_{\bar{q},jk}^{(m_1,m_2)}(q_l',\mathbf{x}\mathbf{A}_l) = [\mathbf{x}\mathbf{A}_l]_j^{m_1}[\mathbf{x}\mathbf{A}_l]_k^{m_2}\delta_{\bar{q},q_l'} \overset{(a)}{=} [\mathbf{x}^{m_1}\mathbf{A}_l]_j[\mathbf{x}^{m_2}\mathbf{A}_l]_k\delta_{\bar{q},q_l'}, \quad (38)$$

$$\psi_{\bar{q},jk}^{(s_1,s_2)}(q_l',\mathbf{x}\mathbf{A}_l) = e^{s_1[\mathbf{x}\mathbf{A}_l]_j + s_2[\mathbf{x}\mathbf{A}_l]_k}\delta_{\bar{q},q_l'}, \quad (39)$$

$$\delta_{\bar{q},q_l'}\delta_{q_l,q} = \begin{cases}\delta_{q_l,q}, & l\in\mathcal{L}_{\bar{q}}',\\ 0, & \text{otherwise}\end{cases}, \quad \delta_{\bar{q},q}\delta_{q_l,q} = \begin{cases}\delta_{\bar{q},q}, & l\in\mathcal{L}_{\bar{q}},\\ 0, & \text{otherwise},\end{cases} \quad (40)$$

where step (a) in (38) follows from the fact that $\mathbf{A}_l$ has no more than a single 1 in a column. Thus, we have

$$\theta_{\bar{q},jk}^{(m_1,m_2)}(q,\mathbf{x}) = \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}[\mathbf{x}^{m_1}\mathbf{A}_l]_j[\mathbf{x}^{m_2}\mathbf{A}_l]_k\delta_{q_l,q} - x_j^{m_1}x_k^{m_2}\delta_{\bar{q},q}\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (41)$$

$$\theta_{\bar{q},jk}^{(s_1,s_2)}(q,\mathbf{x}) = \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}e^{s_1[\mathbf{x}\mathbf{A}_l]_j + s_2[\mathbf{x}\mathbf{A}_l]_k}\delta_{q_l,q} - e^{s_1x_j+s_2x_k}\delta_{\bar{q},q}\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}. \quad (42)$$

Finally, the system of differential equations in (7) and (8) can be derived by applying Dynkin's formula [17] to each test function and its associated extended generator. In particular, the Dynkin's formula can be expressed as

$$\frac{\mathrm{d}\mathbb{E}[\psi(q(t),\mathbf{x}(t))]}{\mathrm{d}t} = \mathbb{E}[L\psi(q(t),\mathbf{x}(t))]. \quad (43)$$

Hence, from (3), (31) and (34), we get

$$\begin{aligned}\dot{v}_{\bar{q},jk}^{(m_1,m_2)}(t) &= \mathbb{E}[L\psi_{\bar{q},jk}^{(m_1,m_2)}(q(t),\mathbf{x}(t))],\\ &= \mathbb{E}[\frac{\partial \psi_{\bar{q},jk}^{(m_1,m_2)}(q(t),\mathbf{x}(t))}{\partial \mathbf{x}(t)}\mathbf{1}^{\mathrm{T}}] + \underbrace{\mathbb{E}[\theta_{\bar{q},jk}^{(m_1,m_2)}(q(t),\mathbf{x}(t))]}_{A},\\ &= m_1v_{\bar{q},jk}^{(m_1-1,m_2)}(t) + m_2v_{\bar{q},jk}^{(m_1,m_2-1)}(t) + A, \end{aligned} \quad (44)$$

where A is given by

$$A = \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}[\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{\bar{q}_l}^{(m_1,m_2)}(t)\mathbf{A}_l]_{j,k} - v_{\bar{q},jk}^{(m_1,m_2)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}. \quad (45)$$

Substituting (45) into (44) yields

$$\dot{v}_{\bar{q},jk}^{(m_1,m_2)}(t) = m_1v_{\bar{q},jk}^{(m_1-1,m_2)}(t) + m_2v_{\bar{q},jk}^{(m_1,m_2-1)}(t) + \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}[\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{\bar{q}_l}^{(m_1,m_2)}(t)\mathbf{A}_l]_{j,k} - v_{\bar{q},jk}^{(m_1,m_2)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}. \quad (46)$$

Further, from (4), (32) and (35), we get

$$\begin{aligned}\dot{v}_{\bar{q},jk}^{(s_1,s_2)}(t) &= \mathbb{E}[L\psi_{\bar{q},jk}^{(s_1,s_2)}(q(t),\mathbf{x}(t))],\\ &= \mathbb{E}[\frac{\partial \psi_{\bar{q},jk}^{(s_1,s_2)}(q(t),\mathbf{x}(t))}{\partial \mathbf{x}(t)}\mathbf{1}^{\mathrm{T}}] + \mathbb{E}[\theta_{\bar{q},jk}^{(s_1,s_2)}(q(t),\mathbf{x}(t))],\\ &= (s_1+s_2)v_{\bar{q},jk}^{(s_1,s_2)}(t) + \mathbb{E}[\theta_{\bar{q},jk}^{(s_1,s_2)}(q(t),\mathbf{x}(t))], \end{aligned} \quad (47)$$

where $\mathbb{E}[\theta_{\bar{q},jk}^{(s_1,s_2)}(q(t),\mathbf{x}(t))] =$

$$\sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}\mathbb{E}\Big[e^{s_1[\mathbf{x}(t)\mathbf{A}_l]_j + s_2[\mathbf{x}(t)\mathbf{A}_l]_k}\delta_{q_l,q(t)}\Big] - \mathbb{E}[e^{s_1x_j(t)+s_2x_k(t)}\delta_{\bar{q},q(t)}]\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)} = \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)} [\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{\bar{q}_l}^{(s_1,s_2)}(t)\mathbf{A}_l]_{j,k} + c_{\bar{q},jk}(t) - v_{\bar{q},jk}^{(s_1,s_2)}(t)\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}, \quad (48)$$

such that $\sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}\mathbb{E}\Big[e^{s_1[\mathbf{x}(t)\mathbf{A}_l]_j + s_2[\mathbf{x}(t)\mathbf{A}_l]_k}\delta_{q_l,q(t)}\Big] = c_{\bar{q},jk}(t)$ when $[\mathbf{x}(t)\mathbf{A}_l]_j = 0$ and/or $[\mathbf{x}(t)\mathbf{A}_l]_k = 0$, and $c_{\bar{q},jk}(t)$ is given by (9). Substituting (48) into (47) yields

$$\dot{v}_{\bar{q},jk}^{(s_1,s_2)}(t) = \Big[s_1+s_2-\sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}\Big]v_{\bar{q},jk}^{(s_1,s_2)}(t) + \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}[\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{\bar{q}_l}^{(s_1,s_2)}(t)\mathbf{A}_l]_{j,k} + c_{\bar{q},jk}(t). \quad (49)$$

The system of differential equations in (7) and (8) can be obtained by gathering the equations in (46) and (49), and writing them in a matrix form. ■

B. *Proof of Theorem 1*

We start the proof by writing the differential equations in Lemma 1 in a combined way as follows:

$$\dot{\mathbf{v}}^{(m_1,m_2)}(t) = m_1\mathbf{v}^{(m_1-1,m_2)}(t) + m_2\mathbf{v}^{(m_1,m_2-1)}(t) + \mathbf{v}^{(m_1,m_2)}(t)(\mathbf{B}-\mathbf{D}), \quad (50)$$

$$\dot{\mathbf{v}}^{(s_1,s_2)}(t) = \mathbf{c}(t) + \mathbf{v}^{(s_1,s_2)}(t)\big[\mathbf{B}-\mathbf{D}+(s_1+s_2)\mathbf{I}\big], \quad (51)$$

$\mathbf{v}^{(m_1,m_2)}(t) = \big[\mathrm{vec}\big(\mathbf{V}_0^{(m_1,m_2)}(t)\big)\ \cdots\ \mathrm{vec}\big(\mathbf{V}_{q_{max}}^{(m_1,m_2)}(t)\big)\big]$,
$\mathbf{v}^{(s_1,s_2)}(t) = \big[\mathrm{vec}\big(\mathbf{V}_0^{(s_1,s_2)}(t)\big)\ \cdots\ \mathrm{vec}\big(\mathbf{V}_{q_{max}}^{(s_1,s_2)}(t)\big)\big]$,
$\mathbf{D} = \mathrm{diag}\big[d_0\mathbf{I}_{n^2},\cdots,d_{q_{max}}\mathbf{I}_{n^2}\big]$, $d_{\bar{q}} = \sum_{l\in\mathcal{L}_{\bar{q}}}\lambda^{(l)}$,
$\mathbf{c}(t) = \big[\mathrm{vec}\big(\mathbf{C}_0(t)\big)\ \cdots\ \mathrm{vec}\big(\mathbf{C}_{q_{max}}(t)\big)\big]$,

$$\big[\mathrm{vec}(\mathbf{R}_0)\ \cdots\ \mathrm{vec}(\mathbf{R}_{q_{max}})\big] = \mathbf{v}^{(m_1,m_2)}(t)\mathbf{B}, \quad (52)$$

$$\big[\mathrm{vec}(\hat{\mathbf{R}}_0)\ \cdots\ \mathrm{vec}(\hat{\mathbf{R}}_{q_{max}})\big] = \mathbf{v}^{(s_1,s_2)}(t)\mathbf{B}, \quad (53)$$

such that $\mathrm{vec}(\mathbf{X})$ is the row vector resulting from concatenating the rows of $\mathbf{X}$ into a single long row, $\mathbf{X} = \mathrm{diag}[x_1,\cdots,x_n]$ is a diagonal matrix with $[\mathbf{X}]_{i,j} = x_i\delta_{i,j}, \forall i,j\in 1:n$, $\mathbf{R}_{\bar{q}} = \sum_{l\in\mathcal{L}_{\bar{q}}'}\lambda^{(l)}\mathbf{A}_l^{\mathrm{T}}\mathbf{V}_{q_l}^{(m_1,m_2)}(t)\mathbf{A}_l$, and $\hat{\mathbf{R}}_{\bar{q}} = \sum_{l\in\mathcal{L}'_{\bar{q}}} \lambda^{(l)} \mathbf{A}_l^{\mathrm{T}} \hat{\mathbf{V}}_{q_l}^{(s_1,s_2)}(t)\mathbf{A}_l$. Note that we could construct the expressions in (52) and (53) due to the fact that $\mathrm{vec}(\mathbf{R}_{\bar{q}})$ and $\mathrm{vec}(\hat{\mathbf{R}}_{\bar{q}})$ are linear in $\mathbf{v}^{(m_1,m_2)}(t)$ and $\mathbf{v}^{(s_1,s_2)}(t)$, respectively. Now, in order to figure out the conditions under which (50) is asymptotically stable, we first rewrite (50) in the case where $\dot{\mathbf{v}}^{(1,1)}(t) = \mathbf{0}$ as $t \to \infty$:

$$\bar{\mathbf{v}}^{(1,1)}\mathbf{D} = \bar{\mathbf{v}}^{(0,1)} + \bar{\mathbf{v}}^{(1,0)} + \bar{\mathbf{v}}^{(1,1)}\mathbf{B}, \tag{54}$$

where $\bar{\mathbf{v}}^{(m_1,m_2)} = \lim_{t\to\infty} \mathbf{v}^{(m_1,m_2)}(t)$. A key observation here is that both $\mathbf{B}$ and $\mathbf{D}$ are non-negative matrices. Thus, based on [16, Lemma 2], if $\bar{\mathbf{v}}^{(0,1)} + \bar{\mathbf{v}}^{(1,0)}$ is strictly positive and there exists a positive solution $\bar{\mathbf{v}}^{(1,1)}$ for (54), then all the eigenvalues of $\mathbf{B} - \mathbf{D}$ have strictly negative real parts, and hence (50) is asymptotically stable. Further, when all the eigenvalues of $\mathbf{B} - \mathbf{D}$ have strictly negative real parts, we observe from (51) that there exists $s_0 > 0$ such that for all (s_1, s_2) satisfying $s_1 + s_2 < s_0$, all the eigenvalues of $\mathbf{B} - \mathbf{D} + (s_1 + s_2)\mathbf{I}$ will have strictly negative real parts, which guarantees the asymptotic stability of (51) under the condition that $\mathbf{c}(t)$ converges as $t \to \infty$. Note that $\bar{\mathbf{v}}^{(0,1)}$ and $\bar{\mathbf{v}}^{(1,0)}$ can be constructed using the stationary marginal first moments $\{\mathbf{v}_{\bar{q}}^{(1)}\}$, and according to (9), $\mathbf{c}(t)$ is a function of the marginal MGFs $\{\mathbf{v}_{\bar{q}}^{(s)}(t)\}$ and the distribution of the MC $\{\mathbf{v}_{\bar{q}}^{(0)}(t)\}$ satisfying (12) and (10), respectively. Therefore, based on [16, Theorem 1], we can figure out that both strict positivity of $\bar{\mathbf{v}}^{(0,1)} + \bar{\mathbf{v}}^{(1,0)}$ and convergence of $\mathbf{c}(t)$ as $t \to \infty$ hold when: 1) the MC $q(t)$ is ergodic with distribution $\{\bar{\mathbf{v}}_{\bar{q}}^{(0)} > \mathbf{0}\}$, and 2) there exists a positive fixed point $\bar{\mathbf{v}}_{\bar{q}}^{(1)}, \forall \bar{q} \in \mathcal{Q}$, for the marginal first moment in (11). ∎

C. Proof of Theorem 2

Using the set of transitions in Table I and (18) in Theorem 1, $\{\bar{v}_{\bar{q},12}^{(s_1,s_2)}\}_{\bar{q}\in\mathcal{Q}}$ can be expressed as

$$\lambda\bar{v}_{0,12}^{(s_1,s_2)} = (s_1+s_2)\bar{v}_{0,12}^{(s_1,s_2)} + \mu(\bar{v}_{1,32}^{(s_1,s_2)} + \bar{v}_{2,13}^{(s_1,s_2)}), \tag{55}$$

$$\lambda\bar{v}_{1,12}^{(s_1,s_2)} = (s_1+s_2)\bar{v}_{1,12}^{(s_1,s_2)} + \lambda_1\bar{v}_{0,12}^{(s_1,s_2)}, \tag{56}$$

$$\lambda\bar{v}_{2,12}^{(s_1,s_2)} = (s_1+s_2)\bar{v}_{2,12}^{(s_1,s_2)} + \lambda_2\bar{v}_{0,12}^{(s_1,s_2)}, \tag{57}$$

From (55)-(57), the MGF can be evaluated as

$$\overset{\mathrm{NP}}{M}(s_1,s_2) = \sum_{\bar{q}=0}^{2} \bar{v}_{\bar{q},12}^{(s_1,s_2)} = \frac{\mu+\lambda-(s_1+s_2)}{\mu-(s_1+s_2)}\bar{v}_{0,12}^{(s_1,s_2)}. \tag{58}$$

Thus, we show how $\bar{v}_{0,12}^{(s_1,s_2)}$ can be evaluated in the following. In particular, from (18), we have

$$\mu\bar{v}_{1,32}^{(s_1,s_2)} = (s_1+s_2)\bar{v}_{1,32}^{(s_1,s_2)} + \lambda_1\bar{v}_{0,2}^{(s_2)}, \tag{59}$$

$$\mu\bar{v}_{2,13}^{(s_1,s_2)} = (s_1+s_2)\bar{v}_{2,13}^{(s_1,s_2)} + \lambda_2\bar{v}_{0,1}^{(s_1)}. \tag{60}$$

Now, $\bar{v}_{0,i}^{(s_i)}$ is evaluated from the fixed point of (12) as

$$\bar{v}_{0,i}^{(s_i)} = \frac{\mu\lambda_i\bar{v}_0^{(0)}}{(\mu-s_i)(\lambda-s_i)-\mu\lambda_{-i}}, \tag{61}$$

where $\bar{v}_0^{(0)}$ can be obtained from the fixed point of (10) as $\frac{1}{1+\rho}$. Note that $\bar{v}_{1,32}^{(s_1,s_2)}$ and $\bar{v}_{2,13}^{(s_1,s_2)}$ can be evaluated by substituting $\bar{v}_{0,2}^{(s_2)}$ and $\bar{v}_{0,1}^{(s_1)}$ from (61) into (59) and (60), respectively. Therefore, $\bar{v}_{0,12}^{(s_1,s_2)}$ is obtained by substituting $\bar{v}_{1,32}^{(s_1,s_2)}$ and $\bar{v}_{2,13}^{(s_1,s_2)}$ into (55). The final expression in (19) follows from (58), which completes the proof. ∎

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