A timing game approach for the roll-out of new mobile technologies

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***Abstract*—When adopting a novel mobile technology, a mobile network operator faces the dilemma of determining which is the best time to start the installation of next generation equipment onto the existing infrastructure. In a strategic context, the best possible time for deployment is also the best response to competitors' actions, subject to normative and material constraints and to the customer's adoption curve. We formulate in this paper a finite discrete-time game which captures the main features of the problem for a two-player game played over a prescribed finite horizon. Our numerical results provide insights on the possible optimal tradeoffs for an operator between fixed costs and installation strategies.**

***Index Terms*—technology adoption, timing games, two-player extensive form games.**

I. Introduction

In the telecommunication industry, the roll-out of a new mobile communication technology is a core challenge faced periodically by mobile operators, requiring to upgrade their infrastructure to keep up with ever-increasing traffic demand and key technology upgrades. Indeed, while standardization efforts are already taking off for next generation 6G mobile networks, it is only very recently that 5G technology was introduced into the mobile communication market [1]. Most telecommunication operators are forced to install the 5G technology on their sites in order to face the current ever-increasing traffic demand which has rendered the previous generation of 4G mobile networks inadequate.

A key success element for an operator, in this context, is the *timing* of the upgrades of their infrastructure and the most efficient timing available to them is defined by a competitive setting. Indeed, operators may want to defer the deployment costs for an economic advantage. But an untimely delay with respect to their competitors may represent a risk of future loss, mainly because this might induce their customers to switch to other operators' network. Furthermore, different types of sites have different return on investment and a different customer base, depending on factors such as operators' positioning on the mobile communication market and geographical specifics.

Motivated by the recurring pattern of innovation in the mobile telecommunication industry, in this paper we propose a strategic framework to model the introduction of new wireless technologies and the main tradeoffs therein. Solution concepts from game theory let us identify optimal technology deployment plans. We focus on the case of two competing operators performing their technology upgrades on a given set of sites.

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The problem is formulated in the form of a discrete-time timing game [2], where actions sets available to each player are defined by logistic and normative constraints. For instance, all operators who joined the auction for the allocation of the 5G band spectrum have to follow certain deployment constraints set by the regulator [1]. In turn, the operators' utility is a function of the customers' technology adoption and their distribution over the operator sites, of costs incurred for the technology upgrades and how effective it is the promotion and marketing for the launch of the new technology. Quality of service offered by operators is assumed a piece of public information, as it is constantly monitored by independent authorities (see [3] for France). A player, i.e., an operator, is hence able to track the evolution of the competitors' 5G deployment. On the other hand, upon choosing a 5G service provider, customers are assumed to bind to this operator until the end of the time horizon. To this aim, as a new technology is introduced, the operators use financial subsidies to accelerate its adoption e.g. using discounted offers or advertisement. We assume that such subsidies are budgeted strategically for the whole time horizon and simultaneously adopted by the operators at the beginning the deployment phase.

State of the art. In the literature, several papers studied the investment optimization problem faced by operators aiming to mitigate installation costs of the 5G technology and yet satisfy increasing traffic demand. One approach considers the behaviour of the operator's customers [4], [5]; such models do not take into account the competition among operators. Another approach is to set a cooperative game to determine how investments should be allocated among the operators [6]–[8]. Some models do account for the fact that operators are in competition to serve potential customers [9]–[11], but they neglect the temporal dimension. In the economic theory, the introduction of a new technology belongs to a specific class of models, called *innovation timing games*. This category of games concerns two players selecting the time at which they act [12]–[14]. This standard scenario is concerned, for instance, with the dynamics of an incumbent which defends from a possible entrant in a market [15]. Results for more than two players are derived in [16], [17] under special assumptions. In game theory, games where players pick a time when they act are called *timing games* and can be either in continuous [18] or discrete [2] time. In the literature, discrete timing games have been used for marketing decisions [19]. In this work we focus on a two-player discrete timing game, in which two operators decide when to start the roll-out. Our model also belongs to the class of sequential games [20], in

ISBN 978-3-903176-49-2 © 2022 IFIP

which players act in turns one after another. In the literature, results are only given for specific categories of timing games: stochastic games with one choice [21], Stackelberg games with random-ordered players [22] and games with small discrete time intervals [23].

Main contribution. To the best of our knowledge, this is the first work that introduces a timing game model for the roll-out of mobile technologies. The model can factor in the customers' adoption dynamics, the operators' installation strategies on multiple classes of sites, and constraints related to logistics and regulation. The system model relies on an extensive form game solved with a tailored-made formulation based on a classic resolution method [24]. Numerical simulations provide interesting insight into the strategic approach of the operators and the resulting market shares.

The paper is structured as follows. In Section II we introduce the model and in Section III we provide an example of application. In Section IV we introduce the methods used for model simulations. In Section V we report for numerical results and we interpret them in light of the possible strategies for the operators. Section VI ends the paper.

II. System Model

Modern telecommunication markets have a few mobile operators in strong competition with one another. A significant fraction of their annual turnover is spent on the maintenance and the upgrade of their network. Thus, the main objective of our model is to study strategies to optimize the allocation of operators' resources. Strategies such as prioritising the time of installation on more profitable sites or performing offers and marketing investments have to be optimized in light of competitors' choices, expected returns and regulation constraints. We restrict to the case of two telecommunication operators both acting as rational players. Each player seeks the optimal strategy to maximize their own return. At the moment of introducing a new mobile technology on the market, both operators first invest a promotion and marketing budget, *P&M budget* hereafter, to boost the customer base, i.e., launching marketing campaigns and releasing a fixed number of promotion subscription offers. In our model each operator $i \in N$ chooses independently their P&M budget from a discrete set $s_i \in S_i$. Second, they launch the field deployment campaign for the new technology, i.e., they schedule when and where to deploy investments on their own *sites*. A site is an area where both operators can install the new technology. Since operators often build sites close to the competitors', we consider perfect overlap of the sites. We assume that there exist few classes of sites, depending on their profitability: metropolitan sites, urban sites, and rural sites. Each of such classes has a different profitability depending, e.g., on the number of subscribers. Operators act on a discrete time horizon $\mathscr{T} = \{1, ..., T\}$. We thus introduce the following parameters:

- $N = \{1, 2\}$, set of players;
- S_i, the P&M budget options for player $i \in N$ (chosen at time $t = 0$);
- $\mathscr{T} = \{1, ..., T\}$, set of time-intervals over which operators act to install the new technology;
- $\mathscr{A}$, site classes.

The possible actions of a player are defined by the variables:

- $s_i \in S_i$, the P&M budget chosen by player i at time 0;
- $A_{it} \in \mathscr{P}(\mathscr{A})$, a subset of *classes of sites* (possibly empty) on which operator $i \in N$ installs the new technology at time $t \in \mathscr{T}$. After the installation, no other intervention is made. We thus denote by $t_{ia} \in \mathscr{T}$ the time at which operator i installs the technology on class $a \in \mathscr{A}$ (in vectorial form $\mathbf{t}_i \in \mathscr{T}^{|\mathscr{A}|}$).

Each operator's schedule is bounded by some constraints:

- *Logistic constraints:* the operator i can invest on a limited number Z_i of classes of sites at each time $t \geq 1$. Thus for every player it holds $|A_{it}| \leq Z_i$;
- *Regulator constraints:* operators must cover a sufficient number of sites within the deadlines fixed by the regulator; before every time t at least R_t classes of sites have to support the new technology. Thus for all players and for all $t \geq 1$ it holds $\sum_{\tau \leq t} |A_{i\tau}| \geq R_t$.

The strategy $(s_i, \mathbf{t}_i) \in S_i \times \mathscr{T}^{|\mathscr{A}|}$ is a choice made by player $i \in N$ on the budget $s_i \in S$ and on the roll-out scheme $\mathbf{t}_i \in \mathscr{T}^{|\mathscr{A}|}$. This choice is subject to the maximisation of the player's objective, which is modeled by her utility function $u_i : S_1 \times S_2 \times (\mathscr{T}^{|\mathscr{A}|})^2 \to \mathbb{R}$. The utility functions depend on multiple parameters, which involve costs, characteristics of the market and adoption dynamics. We defer such analysis to Section III, in which we provide an example of its model.

Strategic framework. Players have conflicting interests, seeking to gain the largest market share, while keeping costs under control. As such, an operator cannot compute their own solution independently from their competitors. We thus identify an equilibrium of the game, i.e., a couple of strategies such that players are satisfied with them if they are both played. We add a further assumption: a player can observe, at time t, the history of actions taken by the other player for $t' < t$, and react accordingly. The choice of the opponent's P&M budget is observable by both players after time $t = 0$.

A convenient model to encode the above assumptions is that of a *game in extensive form* [25], whose mathematical model is based on a *game tree*. In a game tree, arcs outgoing from a node represent the possible choices available to a player. The sequence of nodes and arcs represent the sequence of choices taken by the players and the moments at which players act. A leaf or terminal node h reached after both players have performed T actions has no outgoing arcs and represents an outcome of the game. We denote with H the set of terminal nodes $h \in H$; occasionally we will use the same notation to identify the unique path that leads to a terminal node. Every outcome $h \in H$ is assigned a pair of values $(u_1(h), u_2(h)) \in \mathbb{R}^2$ which correspond to the values assigned by each player to such outcome. The higher the value of $u_i(h)$, the higher the value a player assigns to the combination of actions that leads to the final node h. Every combination of strategies $(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \in S_1 \times S_2 \times (\mathscr{T}^{|\mathscr{A}|})^2$ leads to a unique outcome $h \in H$. For our system the *game tree* is defined as the Service providers (SP) game (cf. Definition 1).

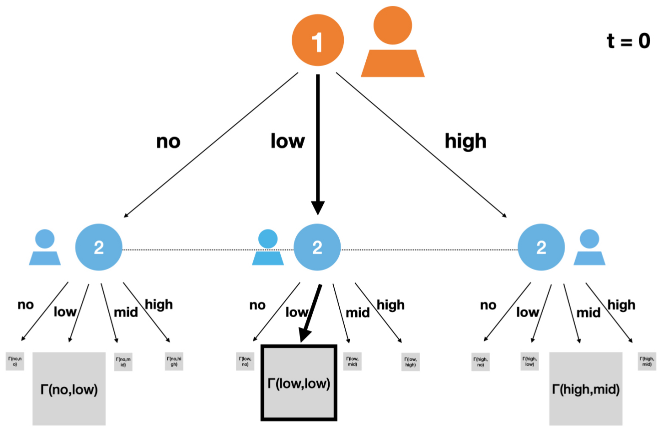

Fig. 1. The P&M budgets are picked first. Both players choose a *low* budget, i.e. $(s_1, s_2) = (\text{low}, \text{low})$, leading thus to game $F = \Gamma(\text{low}, \text{low})$. In Figure some $\Gamma(s_1, s_2)$ are highlighted as example.

	no	low	mid	high
no	$M(\text{no}, \text{no})$	$M(\text{no}, \text{low})$	$M(\text{no}, \text{mid})$	$M(\text{no}, \text{high})$
low	$M(\text{low}, \text{no})$	$\mathbf{M}(\textbf{low}, \textbf{low})$	$M(\text{low}, \text{mid})$	$M(\text{low}, \text{high})$
high	$M(\text{high}, \text{no})$	$M(\text{high}, \text{low})$	$M(\text{high}, \text{mid})$	$M(\text{high}, \text{high})$

TABLE I
MATRIX REPRESENTATION OF THE FIRST PHASE.

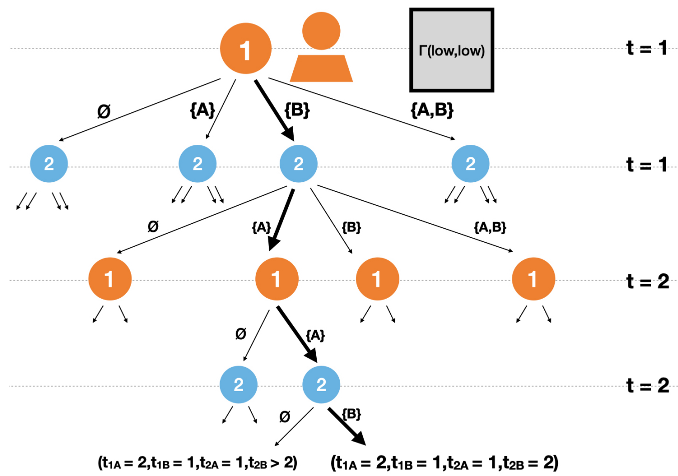

Fig. 2. After the P&M budgets, players choose sequentially to add or not the technology to sites of class A and B at each round $t \in \mathscr{T} = \{1, 2, \ldots, T\}$.

Definition 1 (Service providers (SP) game): The service providers game $\langle N, S_1, S_2, \mathscr{A}, T, u\rangle$ is an extensive form game with two players $N = \{1, 2\}$ competing over set of classes of sites $\mathscr{A}$ in which:

- at the root vertex both players choose at the same time $t = 0$ and independently the P&M budgets $s_1 \in S_1$ and $s_2 \in S_2$;
- the players act in sequence at every round $t \geqslant 1$, starting from player 1. At every step they can decide on which subset of classes of sites $A_{1t} \in \mathscr{P}(\mathscr{A})$ and $A_{2t} \in \mathscr{P}(\mathscr{A})$ install the new technology, given the logistic and regulator constraints;
- after T rounds the game ends; the actions chosen at each round are evaluated by utility functions u_i hereafter defined;
- players' utility function $u = (u_1, u_2)$, where $u_i : S_1 \times S_2 \times (\mathscr{T}^{|\mathscr{A}|})^2 \to \mathbb{R}$.

Example. Figures 1 and 2 show a representation of a game with the following properties:

- $\mathscr{A} = \{A, B\}$, two classes of sites.
- $\mathscr{T} = \{1, 2\}$, horizon of two time-intervals.
- $S_1 = \{\text{no}, \text{low}, \text{high}\}$ and $S_2 = \{\text{no}, \text{low}, \text{mid}, \text{high}\}$ sets of possible P&M budgets.

Overall, we can distinguish two phases of the game. In the *first phase* both players choose their P&M budget for the launch of the technology (cf. Section II-A). In the *second phase* players act in sequence one after another and choose on which sites install the new technology (cf. Section II-B).

A. First phase: P&M budget choice

Both players pick independently their P&M budget $s_i \in S_i$, $i = 1, 2$ (cf. Figure 1). Once budgets (s_1, s_2) are revealed, the players proceed in choosing the timing of their investments. We denote this second part of the game $\Gamma(s_1, s_2)$, which is represented in Figure 2 for our example. In Section II-B we discuss in detail how to compute the timing of investments as a solution $(\mathbf{t}_1, \mathbf{t}_2) \in (\mathscr{T}^{|\mathscr{A}|})^2$ of game $\Gamma(s_1, s_2)$. The outcome $(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \in S_1 \times S_2 \times (\mathscr{T}^{|\mathscr{A}|})^2$ is evaluated by the utility function $u(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \in \mathbb{R}^2$. We define map $M : S_1 \times S_2 \in \mathbb{R}^2$ which evaluates for every pair (s_1, s_2) the solution of the game $\Gamma(s_1, s_2)$: $M(s_1, s_2) := u(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2)$, where $(\mathbf{t}_1, \mathbf{t}_2) \in (\mathscr{T}^{|\mathscr{A}|})^2$ solves $\Gamma(s_1, s_2)$. Let us suppose now to have already computed the solutions of the second phase $M(s_1, s_2)$ for every $(s_1, s_2) \in S_1 \times S_2$ with the method given in Section II-B. Table I shows such computation for the game of Figure 1. We would like to find a solution of the game, i.e., a Nash Equilibrium for the corresponding matrix game: a pair of strategies such that an operator, known the strategy of the other one, does not deviate unilaterally.

Definition 2 (SP game, solution first phase): Given a SP game $\langle N, S_1, S_2, \mathscr{A}, T, u\rangle$ and its correspondent matrix $M = (M_1, M_2) : (s_1, s_2) \mapsto u(s_1, s_2, \sigma(s_1, s_2))$, with (s_1, s_2) chosen at time $t = 0$ and $\sigma(s_1, s_2) \in (\mathscr{T}^{|\mathscr{A}|})^2$ the optimal installation times chosen at times $t \geq 1$, we say $(\bar{s}_1, \bar{s}_2) \in S_1 \times S_2$ is an equilibrium if for all $s_1 \in S_1$ and $s_2 \in S_2$ we have:

$$M_1(\bar{s}_1, \bar{s}_2) \geq M_1(s_1, \bar{s}_2) \quad \text{and} \quad M_2(\bar{s}_1, \bar{s}_2) \geq M_2(\bar{s}_1, s_2).$$

B. Second phase: timing of network investments

Let both operators have chosen their P&M budgets $(s_1, s_2) \in S_1 \times S_2$. The second phase of the game is represented by the subtree $\Gamma(s_1, s_2)$ of Figure 2. At the root the first player chooses a subset of sites where to invest at time $t = 1$. The choices are represented by the outgoing arcs. At the following node the second player chooses their own site class for time $t = 1$. Their actions lead to a node at which the first player chooses the installations to be made at time $t = 2$, and so on until the planning for each time interval is assigned. The two operators know which choice of the P&M

budget is made by the other operator and then act in sequence one after another. At every time $t \in \mathscr{T}$ each operator $i \in N$ chooses a subset of classes of sites where to invest. Formally, the subtree $\Gamma(s_1, s_2)$ is an extensive form game, identified by the tuple $\Gamma(s_1, s_2) = \langle N, H, u \rangle$ from now on, where H is the set of terminal nodes of the game tree. With some notations' abuse $h \in H$ identifies either a terminal node or the path that leads to it from the root. Note that along a path a player can pick a site class only once (installation can be done only once).

In the example of Figure 2 path $h = \{\{B\}, \{A\}, \{A\}, \{B\}\}$ is highlighted: at time $t = 1$ the first operator installs the new technology on class B, the second operator installs it on class A, while at time $t = 2$ the two operators do the opposite.

Definition 3 (SP game, second phase): Given a SP game $\langle N, S_1, S_2, \mathscr{A}, T, u \rangle$ and a couple of P&M budget choices $(s_1, s_2) \in S_1 \times S_2$ we consider the extensive form game $\Gamma(s_1, s_2) = \langle N, H, u \rangle$, whose paths $h = (h_{it})_{i \in N, t \in \mathscr{T}} \in H$ have the following properties:

- for all $(i, t) \in N \times \mathscr{T}$ we have $h_{it} \in \mathscr{P}(\mathscr{A})$;
- for all $i \in N$ and for all $t, t' \in \mathscr{T}$ with $t \neq t'$ we have $h_{it} \cap h_{it'} = \emptyset$.

Later in the section we introduce the solution of a game, which is a specific terminal node $h \in H$. We represent such node with a vector $(t_{ia})_{i \in N, a \in \mathscr{A}}$, which indicates the time $t_{ia} \in \mathscr{T}$ at which operator i installs the new technology on class a.

In the tree representing the game (see Figure 2) a step is represented by a node and the actions that follows by the outgoing arcs. A *subgame* describes the part of the game that follows a given step: it corresponds to the part of the tree that follows a node. Formally, we identify the node with the vector of actions that leads to it. Such vector is a prefix $\overline{h}$ of the vector that corresponds to a path $h \in H$. A subgame is the collection of the paths that share the same prefix. Given path $h \in H$, we write $h = \overline{h} + h'$ to show that $\overline{h}$ is a prefix and h' is the rest of the path.

Definition 4 (subgame): Given a game $\Gamma(s_1, s_2) = \langle N, H, u \rangle$ and a prefix $\overline{h}$, a subgame is a game $\overline{\Gamma} = \langle N, \overline{H}, \overline{u} \rangle$ such that:

- $\overline{H}$ is the set of paths in H sharing prefix $\overline{h}$: $\overline{H} = \{h' : \exists h \in H, h = \overline{h} + h'\}$;
- The utility function in $\overline{\Gamma}$ corresponds to the ones in Γ, i.e. for all $h' \in \overline{H}$ we have that: $\overline{u}(h') = u(\overline{h} + h')$.

As defined before, we would like to identify a solution of the game under the assumption that every operator at time t can observe the actions taken by the other player at time $t' < t$. This corresponds to require the solution to be an equilibrium for every subgame, i.e., to be a *subgame perfect equilibrium* (SPE) [20]. Operators can forecast future actions, starting from the bottom of the tree, since they both know that they choose the actions that maximise their utility, proceeding thus by *backward induction* [20]. If an operator is on a subgame corresponding to a leaf of the game tree, they pick an action that maximises their utility. In recursive manner, in any parent subgame an operator can identify the actions played subsequently and also identify the action that leads to the best outcome for them. The solution identified by this algorithm corresponds to a SPE. A generic finite extensive form game with perfect information has a unique SPE equilibrium [20] and it is generated by the backward induction algorithm [25].

The choice of the order of the operators can be arbitrary. However, while in reality they act simultaneously, when both players can quickly adjust their strategy accordingly, the outcome of the game undergoes a small perturbation, as we see in simulations.

III. Subscriber dynamics and operator utility

So far we have not discussed the operators' utility function $u_i : S_1 \times S_2 \times (\mathscr{T}^{|\mathscr{A}|})^2 \to \mathbb{R}$. As described next, it highly depends on the adoption dynamics, i.e., how many customers switch to the new mobile technology at each time unit. To characterize the utility function we fix some assumptions: 1) following [4], every customer decides to switch to the new technology at a given time t and sticks to the choice till the end of the time horizon; 2) the quantity of customers switching at every time depends on a given dynamics, which is known a priori by both operators; 3) every customer has a preference over the two operators; if their preferred operator does not offer the new technology at time t they subscribe to the other operator if it offers it, otherwise they wait for one of them to offer it; 4) an operator can acquire only a limited amount of new customers per time interval; 5) once a customer subscribes for an operator, it sticks to it. Operators choose a pair of P&M budgets $(s_1, s_2) \in S_1 \times S_2$; such choices influence the potential market of customers willing to switch to the new technology.

We consider first the scenario where all sites belong to the same class, and we then extend the analysis to the case where sites belong to different ones.

A. Single-class model

For every $t \in \mathscr{T}$, the percentage of customers adopting the new mobile technology is identified by $\{y_t\}_{t \in \mathscr{T}}$, subject to the condition that $\sum_{t \in \mathscr{T}} y_t = 1$ because all customers eventually switch to the new technology. Hence, $\{y_t\}$ can be seen as the time-step increment $y_t := y_{adopt}(t + 1) - y_{adopt}(t)$ of the adoption curve $y_{adopt}(t)$, which is assumed to be non-decreasing concave and $y_{adopt}(\infty) = 1$. A model often used in the literature [27] for adoption curves is $y_{adopt}(t) = 1 - e^{-\lambda t}$, where the parameter $\lambda > 0$ determines the adoption speed.

Preferences over the offers proposed by operators 1 and 2 are a function of the P&M budgets. They are distributed with same proportion $p_1(s_1, s_2)$ and $p_2(s_1, s_2) = 1 - p_1(s_1, s_2)$ for all $t \in \mathscr{T}$. We fix $p_1(s_1, s_2) = \frac{p_{01} + k \cdot s_1}{1 + k \cdot (s_1 + s_2)}$ and $p_2(s_1, s_2) = \frac{p_{02} + k \cdot s_2}{1 + k \cdot (s_1 + s_2)}$, where $p_{01}, p_{02} \in [0, 1]$ with $p_{01} + p_{02} = 1$, the preference of a customer for operator 1 or 2, respectively, if $s_1 = s_2 = 0$. Here, k is a parameter that weights the influence of the P&M budget on the customers. We recall that y_t is the potential market at time t. In order to capture these customers the operators have to install the new technology; once their sites are equipped with the new technology, we assume that mobile operators can acquire at most $\tau > 0$ share of new customers per time slot. This constraint models the fact that potential customers resolve for the technology little by little and not all at once. Let $\alpha_i(t) := \alpha_i(t; s_1, s_2, \mathbf{t}_1, \mathbf{t}_2)$ the revenue for player i which can be ascribed to customers acquired up to time $t \in \mathscr{T}$, respectively, under a given multistrategy $(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2)$ (in case of a single site (s_1, s_2, t_1, t_2)). Once acquired, they are retained until the end of the horizon: the revenue generated by the final market share is $\alpha_i(T)$.

On the other hand, operators face the costs related to the installation of the new technology and the subsidies. P&M budgets $s_i \in S_i$ are constant fixed costs. Conversely, $c_i(t) > 0$ is the installation cost at time $t \in \mathscr{T}$: it is discounted to account for the depreciation since installation time t and lower maintenance costs over the period. Player i installing at time $t = 1$ incurs in cost $c_i(1) = c_0 > 0$, whereas we assume $c(T) = 0$. A linear discounted cost model sets $c_i(t) = c_0 \cdot \frac{T-t}{T-1}$. Finally, the utility for player i writes

$$u_i(s_1, s_2, t_1, t_2) = \alpha_i(T) - c_i(t_i) - s_i.$$

Summing up, the game is determined by parameters:
- $\lambda > 0$: adoption speed;
- $\tau > 0$: maximal share of customers that can be acquired in a time interval;
- $S_1, S_2 \subset [0, +\infty)$ with $|S_1| < \infty$ and $|S_2| < \infty$: discrete sets of choices for the P&M budget;
- $p_{0i} \in [0, 1]$, $i = 1, 2$: fraction of customers preferring operator i when no P&M budget is deployed $s_1 = s_2 = 0$;
- $k > 0$, weight of the P&M budget;
- $c_0 > 0$, the installation cost at time $t = 1$.

We now describe how the functions $\alpha_1(t)$ and $\alpha_2(t)$ depend on the strategies chosen by the players among the states $t \in \mathscr{T}$. The system state at time t is also described by the following variables: 1) y_t, the percentage of customers who decide to switch to the new technology at time slot t; 2) t_i, time at which operator i installs the technology; 3) d_t, the demand of customers who want to switch to the new technology at time t and who are not served before time t; 4) d_{it}, the fraction of d_t who prefer the operator i (with $d_{i0} = 0$); 5) r_{it}, customers that operator i can accept at time t. In particular, if an operator i has installed the technology at time t_i, they can accept up to τ customers per interval of time: $r_{it} = \tau \cdot \mathbb{1}\{t \geq t_i\}$.

The demand at time t is updated with the new costumers. All the customers that are added to the demand at time t are y_t. Given $p_i := p_i(s_1, s_2)$ the fraction of customers who prefer operator i, we have that the number of customers who prefer i at time t are $p_i \cdot y_t$, i.e. $\forall i \in N$: $d_{it} \leftarrow d_{i,t-1} + p_i \cdot y_t$.

Up on their demand and on their supply, the operators add their customers. The operator i can absorb the unfulfilled demand of its competitor j. The dynamics of customers acquired by operator i is thus governed by equation $\alpha_i(t+1) = \alpha_i(t) + \min(d_{it}, r_{it}) + [\min(d_{jt} - r_{jt}, r_{it} - d_{it})]^+$ where $i \neq j$. Finally, the demand at time $t + 1$ is set at $d_{i,t+1} \leftarrow [d_{it} - r_{it} - [r_{jt} - d_{jt}]^+]^+$ since costumers acquired at time t are removed from their demand at time $t + 1$.

B. Multi-class model

Let us consider a model for more than one class of sites $|\mathscr{A}| > 1$. We assume further that every customer is served on sites belonging to same class, and thus introduce the parameter $\omega_a \in (0, 1)$, the percentage of users that are served on sites of class $a \in \mathscr{A}$. Accordingly, we assign to every parameter an index referring to the relative class. The potential market $y_{a,t}$ for class $a \in \mathscr{A}$ is subject to the condition $\sum_{t \in \mathscr{T}} y_{a,t} = \omega_a \cdot y_t$. The utility function for the multi-class case differs from the single-class by the fact that it sums the customers acquired on every class of sites and the costs of the respective installations. For every player $i \in N$ the utility writes

$$u_i(s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) = \sum_{a \in \mathscr{A}} \alpha_{i,a}(T) - \sum_{a \in \mathscr{A}} c_i(t_{i,a}) - s_i$$

where the first summation is the revenue calculated across all sites, and $\alpha_{i,a}$ and $c_i(t_{i,a})$ are respectively the utility and the cost for player i for sites of class $a \in \mathscr{A}$.

IV. Upper bounds

As described in Section II, the SPE determined by backward induction requires the explicit construction of the game tree, whose size is polynomial in the time horizon but exponential in the number of site classes. An efficient method for the solution of the game is the one introduced by Von Stengel [24], solving the equilibrium of extensive-form games via a bilevel optimization problem. Compared to the SPE proposed before, the solution found provides an upper bound to any Nash equilibrium of the timing game, and so to the SPE as well. The method is defined around the concept of *sequence*. A sequence is a vector of consecutive actions played by the same player. Let us consider a game with $T = 2$ and a path $h = (h_{11}, h_{21}, h_{12}, h_{22}) \in H$, as for instance $h = \{\{B\}, \{A\}, \{A\}, \{B\}\}$ of Figure 2. The actions h_{11} and h_{12} are played by the first player, while h_{21} and h_{22} are played by the second player. Sequences of the first player are $seq_1 = (h_{11})$ and $seq_1 = (h_{11}, h_{12})$ while sequences of the second player are $seq_2 = (h_{21})$ and $seq_2 = (h_{21}, h_{22})$. Formally, for every player $i \in N$ we consider the set of their sequences $\Sigma_i = \{seq_i = (h_{i\tau}), \tau \leq t, t \in \mathscr{T}, h \in H\}$. Let $x \in \{0,1\}^{|\Sigma_1|}$ and $y \in \{0,1\}^{|\Sigma_2|}$ the vectors which define the probability for a sequence to be played. We confine to pure strategies, i.e., either $x_{seq_1} = 1$ if sequence seq_1 is played or $x_{seq_1} = 0$ if sequence seq_1 is not played. Every path h that leads to a leaf and the two corresponding sequences $(seq_1^h, seq_2^h) \in \Sigma_1 \times \Sigma_2$ are evaluated by the utility function $u(h)$. We define thus the matrices $U^1, U^2 : \Sigma_1 \times \Sigma_2 \to \mathbb{R}$ that map couples of sequences to their utilities: $\forall h \in H$ $U^1_{seq_1^h, seq_2^h} = u_1(h)$ and $U^2_{seq_1^h, seq_2^h} = u_2(h)$, otherwise $U^1_{seq_1, seq_2} = U^2_{seq_1, seq_2} = 0$. The utilities of the players can be thus written in the form $x^T U^1 y$ and $x^T U^2 y$. Note that the players' sequences are constrained according to the SP game. Indeed, let us suppose that after the path $h = (\{B\}, \{A\})$ the first player has already played $seq_1 = (\{B\})$ and thus can choose among the actions $\{\emptyset, \{A\}\}$. We can write the corresponding sequences $seq_1' = (\{B\}, \emptyset)$ and $seq_1'' = (\{B\}, \{A\})$. If any of these sequences is chosen, then the sequence seq_1 must also be chosen. We thus write $seq_1 = seq_1' + seq_1''$. All such causal constraints $Ex = e$ and $Fy = f$ can be built according to the same principle. Finally, we consider the solution of the following bilevel problem (VS) [24].

$$\begin{aligned} u_1^{VS} = \max_x \quad & x^T U^1 \overline{y} \\ s.t.\ & Ex = e \\ & x \in \{0,1\}^{|\Sigma_1|} \\ & \overline{y} = \max_y \quad x^T U^2 y \\ & \quad s.t. \quad Fy = f \\ & \qquad\quad y \in \{0,1\}^{|\Sigma_2|} \end{aligned} \tag{VS}$$

This bilevel optimization problem is linear in the size of the game [24]. The following result ensures that Problem (VS) provides an upper bound to the utility of any Nash equilibrium.

Theorem 1: Given an extensive-form game $\langle N, H, u \rangle$, the optimal value $u_1^{\text{VS}} \in \mathbb{R}$ of the corresponding optimization problem (VS) and the outcome of a Nash equilibrium $h_{\text{NE}} \in H$, we have:

$$u_1^{\text{VS}} \geq u_1(h_{\text{NE}}).$$

Proof. Every feasible pair (x, y) of VS corresponds to a strategy profile σ such that $u(\sigma) = (x^T U^1 y, x^T U^2 y)$. Also, let $\overline{\sigma}$ correspond to $(x, \overline{y})$: then $u_2(\overline{\sigma}) \geq u_2(\sigma)$ for all $y \in \{0,1\}^{|\Sigma_2|}$ [24]. If σ^{NE} is a Nash equilibrium, be $h_{\text{NE}} \in H$ the outcome corresponding to the pair $(x^{\text{NE}}, y^{\text{NE}})$, then it holds $u_2(\sigma_1^{\text{NE}}, \sigma_2^{\text{NE}}) \geq u_2(\sigma_1^{\text{NE}}, \sigma_2)$ for all σ_2 and thus ${x^{\text{NE}}}^T U^2 y^{\text{NE}} \geq {x^{\text{NE}}}^T U^2 y$ for all $y \in \{0,1\}^{|\Sigma_2|}$. This proves that the pair $(x^{\text{NE}}, y^{\text{NE}})$ is a feasible solution of VS. Given $(x^{\text{VS}}, \overline{y}^{\text{VS}})$ solution of VS, the thesis follows since $u_1^{\text{VS}} = {x^{\text{VS}}}^T U^1 \overline{y}^{\text{VS}} \geq {x^{\text{NE}}}^T U^1 y^{\text{NE}} = u_1(h_{NE})$.

In Section V we apply Theorem 1 to get an upper bound of the utility of the first player in a subgame perfect equilibrium. Indeed, since the subgame perfect equilibrium is a Nash equilibrium, we have that (VS) provides such upper bound.

V. Numerical results

In this section we characterize a scenario for two operators $N = \{1, 2\}$ and three classes of sites $\mathscr{A} = \{A, B, C\}$, which correspond to metropolitan (A), urban (B) and rural sites (C). The roll-out occurs over a period of twelve time intervals $\mathscr{T} = \{1, \ldots, 12\}$, corresponding to twelve quarters, i.e., three years. Due to logistics constraints, operators can install only on a single class of sites per interval of time, i.e., for all $i \in N$ and all $t \in \mathscr{T}$, $|A_{it}| \leq 1$. Moreover, each of them is forced by the regulator to install the new technology on at least one class of sites before the end of the first year ($t \leq 4$), i.e. $R_4 = 1$, and to deploy it everywhere before the end of the game ($t \leq 12$), i.e. $R_{12} = 3$. The first class has a larger customer base than the other two, which are instead of comparable size: $\omega_A = 0.5$, $\omega_B = 0.3$, and $\omega_C = 0.2$. Also, we set $\lambda = 0.1$ and $\tau = \frac{1}{6}$. We use baseline preferences $(p_{01}, p_{02}) = (0.6, 0.4)$ with $k = 3$, that is operator 1 is the incumbent and operator 2 is the entrant. The initial investment cost $c_0 = 0.5$.

To identify the Nash equilibria of the game, we should compute $M(s_1, s_2)$ for every value of the P&M budget $s_1 \geq 0$ and $s_2 \geq 0$. We recall that $M(s_1, s_2)$ is the utility of the unique SPE of the game in extensive form $\Gamma(s_1, s_2)$. Since we can perform such computation only for a finite number of games, we make a suitable discrete selection of the values $s_1 \in S_1$ and $s_2 \in S_2$. The choice of $S_1 = \{0, 0.1, 0.2\}$ and $S_2 = \{0, 0.1, 0.2, 0.3\}$ is justified hereafter. We refer to them also as respectively $\{\text{no}, \text{low}, \text{mid}\}$ and $\{\text{no}, \text{low}, \text{mid}, \text{high}\}$ budget for clarity.

We suppose that large values of s_1 and s_2 are not chosen by the operators. Indeed, if the operators invest too much in the P&M budget, they cannot pay off their investment. We would like thus to exclude those strategies that are not played by the operators. Formally, a strategy for the first player s_1 is said to be *dominated* if there exists another strategy s_1' that provides better utility, no matter what the second player chooses as their strategy.

We first explore the best responses for different budget choices and under different customers' baseline preferences over the two operators. Figure 3a) shows the best response of the entrant operator, where $s_2 \in [0, 0.3]$, when the incumbent operator does not spend any P&M budget for the launch of the new technology, i.e., $s_1 = 0$. The utilities of the two players evolve, given different choices of the budget of the second operator. The best response for the entrant operator is to play $s_2 = 0.03$ because it maximises the utility of the second player $M_2(s_1, s_2) = 0.25$. We can observe that the utility of the incumbent $M_1(s_1, s_2)$ is generically decreasing in the P&M budget of the entrant s_2.

We will now prove that some strategies are dominated, so that they can be excluded to compute Nash equilibria. Figure 3b) shows how the utility of the first player evolves when they choose $s_1 = 0$ and the second player chooses $s_2 \geq 0$. Strategies dominated by $s_1 = 0$ can be computationally expensive to calculate for many values $s_1 > 0$. However, we can provide an upper bound $u_1^{VS}(s_1, s_2; s_1)$ of function $u_1(s_1, s_2; s_1)$ by using the method introduced in Section IV. The solution of (VS) provides an upper bound to the utility of the first player at the SPE. We compute it for different values of $s_1 \geq 0$. Figure 3b) shows the graphs of $u_1^{VS}(s_1, s_2; s_1)$ for $s_1 = \{0.1, 0.2, 0.3, 0.4, 0.5\}$. We find that the upper bound of the utility of the first player given $s_1 \geq 0.3$ is always lower than the utility given by $s_1 = 0$, i.e. $u_1^{VS}(s_1, s_2; s_1 \geq 0.3) < u_1(s_1, s_2; s_1 = 0)$ for all $s_2 \in S_2$. Since $u_1^{VS}(s_1, s_2; s_1 \geq 0.3) \geq u_1(s_1, s_2; s_1 \geq 0.3)$, we conclude that any strategy $s_1 \geq 0.3$ does not provide a better utility to the first player than $s_1 = 0.0$, no matter what the second player chooses. We thus exclude any value of $s_1 \geq 0.3$ for the first player. By performing a similar analysis on the second player we can thus limit the choice of the discrete sets to $S_1 = \{0, 0.1, 0.2\}$ and $S_2 = \{0, 0.1, 0.2, 0.3\}$.

The choice of the P&M budget changes the customers' baseline preference for operators. Figure 4 shows how the market shares are distributed over the classes of sites as a function of the resulting customers' preference. As we observe there, in some cases the investment can become profitable only if an operator obtains the monopoly over a class of sites. We recall that the regulator's constraint forces both operators to install early on at least one class of sites. Where

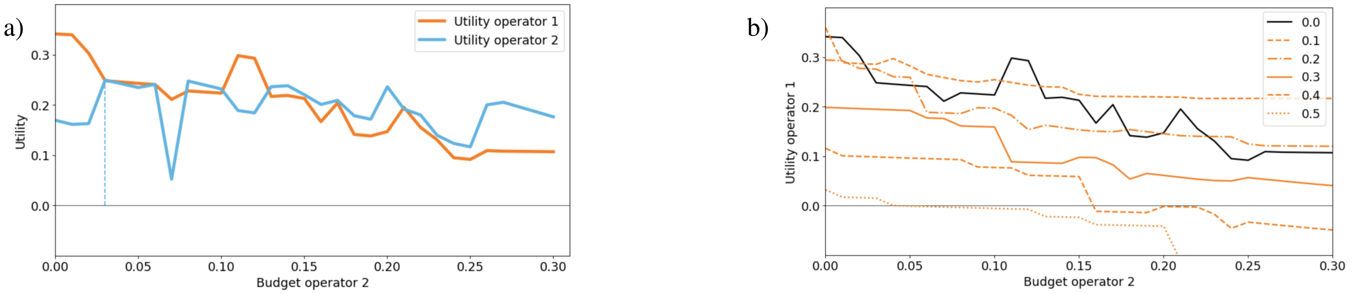

Fig. 3. a) Evolution of $M(s_1, s_2)$ with $s_1 = 0.0$ and $s_2 \in [0.0, 0.3]$. b) Utility of the first player $M_1(s_1, s_2)$ for $s_1 = 0.0$ and $s_2 \in [0.0, 0.3]$, and its upper bound $u_1^{VS}(s_1, s_2)$ with $s_1 = \{0.1, 0.2, 0.3, 0.4, 0.5\}$ and $s_2 \geq 0.0$. The strategies $s_1 \in \{0.3, 0.4, 0.5\}$ are dominated by the strategy $s_1 = 0.0$.

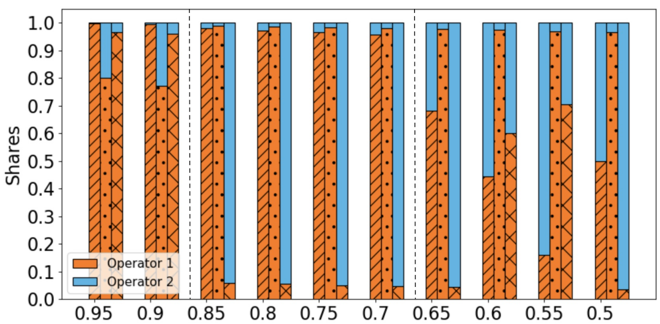

Fig. 4. Different share distribution for different values of $p_1 \in \{0.95, 0.9, 0.85, 0.8, 0.75, 0.7, 0.65, 0.6, 0.55, 0.5\}$. The three columns represent the three classes of sites $\{A, B, C\}$, respectively with hatch '/','·','x' for the first operator.

to invest early determines the possibility to create a monopoly on a certain class of sites and/or prevent the opponent to get one. For instance, if the incumbent has almost the monopoly, i.e., $p_1 \geq 0.9$, they find it profitable to install early on all the classes. If the incumbent is strongly dominant, i.e. $0.7 \leq p_1 < 0.9$, they do not find it profitable to invest on class C, which is left as a monopoly to Operator 2. If Operator 1 is weakly dominant, i.e. $0.5 \leq p_1 < 0.7$, they get the monopoly on class B, but they are also forced to share the customer base on class A and even leave the monopoly on class C to Operator 2. The variability of the solutions on this case with 3 classes suggests that their classification is not trivial. The analysis of such classification and how parameters influence it is left to future works.

The solutions of the game are reported in Figure 5. More precisely Figure 5a) reports on the solutions of the timing game for each P&M budget pair, whereas Table 5b) reports on the corresponding attained utility. From Figure 5a) we observe how different P&M budgets – by influencing the preferences of customers – lead to different investment plans. Also, they induce different shares of the customer base over different classes of sites as observed by comparing the possible scenarios of Figure 4.

For instance, if both operators do not invest any P&M budget for the launch of the new technology, namely $(s_1, s_2) = (\text{no}, \text{no})$, they both deploy from the start ($t = 1$) in rural areas (class C). Afterwards, the first operator (the incumbent) installs in urban areas (B) and finally both operators install in metropolitan areas (A). Operators roughly split evenly customers in metropolitan areas (A) and in rural areas (C), while the incumbent obtains the monopoly in urban areas (B), i.e., where they install first.

We consider now the case when the incumbent increases the P&M budget: $(s_1, s_2) = (\text{low}, \text{no})$. In this case the incumbent installs immediately in metropolitan areas (A), i.e., at time $t = 1$. They thus obtain a monopoly over this class of sites. The entrant reacts by installing in rural areas (C) at time $t = 2$ and takes over those sites by getting a monopoly there. At time $t = 3$ the incumbent operator installs in urban areas (B) and gets all the market share for those sites. Finally, in such case metropolitan (A) and urban areas (B) are almost completely assigned to the first player, while rural areas (C) are monopoly of the second player.

While the existence of a Nash equilibrium in mixed strategies is guaranteed by theory [26], Figure 5b) indicates that there is only one pure Nash equilibrium (low, mid). We notice that strategy $s_2 = \text{mid}$ is the best response of the entrant operator to any strategy of the incumbent. Since the incumbent operator is a rational player, they know that the entrant operator plays $s_2 = \text{mid}$ and thus chooses $s_1 = \text{low}$ because they increase users' preference to them which maximises their utility. The equilibrium (low, mid) is not Pareto optimal: indeed, both operators attain larger utility by playing (no, low). However, outcome (no, low) is not an equilibrium: if the incumbent does not invest any P&M budget, the entrant gets larger utility by further increasing it. The incumbent thus invests a small P&M budget to avoid the entrant to get an excessive share of the market.

VI. Conclusion

Inspired by the ongoing roll-out of 5G mobile networks we introduced the *Service providers (SP) game*, a timing-game model to determine the optimal strategy for the installation of new mobile technologies. We consider a scenario where two operators have to perform investments over classes of sites with different profitability profiles. The model permits to analyse the dynamics of the roll-out strategies used by the operators accounting for promotion and marketing investments a)

	no	low	mid	high
no	$(9,5,1)$ $\mathbf{0.60}$ $(8,12,1)$ $\mathbf{0.40}$	$(10,4,11)$ $\mathbf{0.46}$ $(10,4,9)$ $\mathbf{0.54}$	$(12,9,4)$ $\mathbf{0.37}$ $(2,9,12)$ $\mathbf{0.63}$	$(12,11,3)$ $\mathbf{0.32}$ $(2,8,12)$ $\mathbf{0.68}$
low	$(1,3,12)$ $\mathbf{0.69}$ $(11,12,2)$ $\mathbf{0.31}$	$(3,2,1)$ $\mathbf{0.56}$ $(2,11,12)$ $\mathbf{0.44}$	$(10,4,11)$ $\mathbf{0.47}$ $(10,4,9)$ $\mathbf{0.53}$	$(9,12,4)$ $\mathbf{0.41}$ $(8,3,12)$ $\mathbf{0.59}$
mid	$(2,4,12)$ $\mathbf{0.75}$ $(11,12,3)$ $\mathbf{0.25}$	$(1,3,12)$ $\mathbf{0.63}$ $(2,12,1)$ $\mathbf{0.37}$	$(9,3,8)$ $\mathbf{0.55}$ $(1,12,11)$ $\mathbf{0.45}$	$(10,4,11)$ $\mathbf{0.48}$ $(10,4,9)$ $\mathbf{0.52}$

b)

	no	low	**mid**	high
no	$(0.30, 0.12)$	$(0.26, 0.22)$	$(0.15, 0.24)$	$(0.11, 0.18)$
low	$(0.36, 0.02)$	$(0.16, -0.01)$	$(\mathbf{0.16}, \mathbf{0.12})$	$(0.11, 0.07)$
mid	$(0.30, 0.03)$	$(0.11, -0.07)$	$(0.05, 0.08)$	$(0.07, 0.02)$

Fig. 5. a) Optimal timing: displayed on two lines as (t_{1A}, t_{1B}, t_{1C}) $\mathbf{p_1}$ and (t_{2A}, t_{2B}, t_{2C}) $\mathbf{p_2}$. The baseline preferences (p_1, p_2) influence the strategy for investment timings. For instance, if operators choose $(s_1, s_2) = (\text{no}, \text{no})$ the baseline preferences are $(p_1, p_2) = (0.6, 0.4)$. As we can see from column $p_1 = 0.6$ of Figure 4, the operators split the market in classes A and C, while in class B operator 1 has the monopoly. b) Results of the simulation: utility, displayed in the following order (u_1, u_2). Nash equilibrium is given by (low, mid).

and regulatory constraints. We have provided a solution concept in the form of an equilibrium for the game and a method to compute it. Numerical results show that different strategic patterns depend on the balance between the baseline market share of incumbent and entrant operator. P&M investments may change the share presence of the operators, but higher investments may even result in lesser profits for both.

Discussion and future works. Preliminary results show that the timing of the investments may vary significantly depending on the customers' preferences. A P&M budget able to steer the customers' preferences represents thus a key component of operators' strategic behavior (cf. Figure 4). To this respect the range of the P&M budget can be bounded by exclusion of dominated strategies. This provides a computational advantage, because comparing the utility appearing in the matrix game M is expensive, since it requires to compute the SPE of multiple timing games. We have showed that the technique introduced in Section IV provide an effective bound to the utility that a player attains at the SPE and by computing the upper bound of multiple timing games we could exclude several dominated strategies.

The same model proposed in this paper can be extended to describe roll-out strategies for individual sites. Indeed, an improved granularity of the model would allow a better understanding of the impact of logistic and regulatory constraints at the local level. However, the complexity of the computation of the equilibrium of the game grows exponentially with the number of sites. In future works we intend to explore efficient methods to both reduce the space search and to categorize the optimal investment strategies.

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