THE RESEARCH OF GEOMETRIC CONSTRAINT SOVING BASED ON THE PATH TRACKING HOMOTOPY ITERATION METHOD

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Abstract: Geometric constraint problem is equivalent to the problem of solving a set of nonlinear equations substantially. Nonlinear equations can be solved by classical Newton-Raphson algorithm. Path tracking is the iterative application of Newton-Raphson algorithm. The Homotopy iteration method based on the path tracking is appropriate for solving all polynomial equations. Due to at every step of path tracking we get rid off the estimating tache, and the number of divisor part is less, so the calculation efficiency is higher than the common continuum method and the calculation complexity is also less than the common continuum method.

Key words: Geometric constraint solving, under-constraint, homotopy iteration method, path tracking

1. INTRODUCTION

Geometric constraint solving approaches are made of three approaches: algebraic-based solving approach, rule-based solving approach and graph-based solving approach. One constraint describes a relation that should be satisfied. Once a user defines a series of relations, the system will satisfy the constraints by selecting proper states after the parameters are modified. The idea is named as model-based constraints. Constraint solver is a segment for the system to solve the constraints. In the practical project application, there are three basic needs for the geometric constraint problems: real-time quality (the speed must be fast), integrality (all the solutions must be gained) and stability (a small change from the solutions to the problems it cannot lead to a large change).

Geometric constraint problem is equivalent to the problem of solving a set of nonlinear equations substantially. In constraint set every constraint corresponds to one or more nonlinear equations, all the unattached geometric parameters in the geometric element set constitute the variable set of nonlinear equations. Nonlinear equations can be solved by Newton-Raphson algorithm. When the size of geometric constraint problem is large, the scale of the equation set is very large. Furthermore, the geometric information in the geometric system will not be treated properly and we cannot deal with under- and over-constrained design system well when all the equations must be solved. Accordingly, we can differentiate the geometry constraint method based on algebra method the method based on numerical value and based on symbol. Symbol solving can solve closed formal analytical solution of the problem, it is certainly the most perfect method, but sometimes, its difficulty is also very high. Along with the accretion of the problem's mathematical model, the process of eliminating variable is more and more complicate. [2]

The most universal method is Newton Iteration method, for example, the method proposed by Gossard[3], Assuming the constraint equations $F_i(x_i) = 0$, then the Newton-Raphson iteration formula is

$$X_i^{n+1} = X_i^n - [F_i'(X_i^n)]^{-1} F(X_i^n),$$

Here $\left[F_i'(X_i^n)\right]$ is a Jacobi matrix.

Given the initialized value X_i^0 , we can calculate the new value X_i^1, X_i^2 , ... , X_i^k according to iteration formula, until it satisfy (i) $\|F_i(X_i^k)\| < \delta$, or (ii) $k > N_{max}$. Condition (i): It can indicate that the iteration is convergent in a given time, and success in getting the solution; Condition (ii): indicate that the iteration isn't convergent in a given time, and fails in getting the solution. Generally, the method can get the solution quickly, but it needs a nice initialization value and it is quite sensitive to the initialized value. Using the method, when the initialization value changed a little, it may lead to emanative or converge to the solution that the user doesn't want. The analysis span method that [4] put forward is convergent in quite range, and we can judge if there is no solution in a given range, by the method we can also solve all the solutions in the range if there exist solutions, but there still exist problem choosing appropriate range of the initialization value. Homotopy can overcome this shortcoming[5, 6, 7] in a certain degree. Homotopy is an effective numerical iterative method, it has a strong whole convergence, and it can solve all the solution or a set of the isolated solution of the equations reliably. But the equations from the geometry constraint problem is sometimes under-constraint, so when using the homotopy method to get the solution, we should adopt the homogeneous homotopy method or the coefficient homotopy method, it need to predispose the problem, it not only makes the solving process more complicated, but also needs specific disposal for specific problem, then it decreases the universality of the solving process. For the big scale and under-constraint problem, the calculating efficiency of solving also restricts the practical appliance of the method. [8-13]

Based on the spirit the Homotopy continuum, we put forward a new numerical iterative method that integrates the Homotopy function with the traditional iteration method; we define it as Homotopy iteration method. Similarly with the usual Homotopy method, Homotopy iteration method can effectively solve all numerical solution of the non-linear equations without choosing appropriate initialization value. For the under-degree equation, it not only needn't to predispose such as homogeneous Homotopy or coefficient Homotopy, but also has a higher calculation efficiency and reliability of getting all the solutions.

2. HOMOTOPY ITERATION METHOD

2.1 The Summarization of the Homotopy Method

Homotopy itself is the conception in the Analisissitus. Homotopy algorithm is a whole algorithm that can solve non-linear equations. In 1976, Kellogg and Yorke.Li, solved the global astringency problem of the Homotopy algorithm by differential coefficient topology tools, and proved the theorem's constructive character of Brouwer fixed point, from then on scientists began to restudy the Homotopy method. In several decades, this algorithm became a successfully algorithm and is applied to the Economics, the design of the electronic circuitry, auto control, computer aided design, computer aided manufacture and many other areas and so on.

The main spirit of the Homotopy is that: firstly, choose a simple equation $f(x)=0$, which solution is known, then construct the Homotopy mapping $H(x,t)$ which contains parameter, such that $H(x,0) = f(x), H(x,1) = g(x)$. In a given condition, the solution of the H(x,t) = 0 can define the curve x(t) which starts from the solution of f(x) = 0, $x(0) = x^0$, when t approximates to 1, the curve achieve to the solution x*=x(1) of the g(x) = 0, this is just Homotopy method.

Therefore, Homotopy method can be disparted into two steps: *Step1*. Insert the equations P(x) into a cluster of the equations H(x,t), this is just the reason why we call it as Homotopy. *Step2*. Carry numerical track to the solution curve by the Homotopy-Continuation Method.

Definition1: Homotopy H(x,t) = 0 is defined as a set of equations as follows:

$H(x,t) = c(1-t)^k f(x) + t^k g(x) = 0$, $k \in N$, $c \in C \backslash \{0\}$, $t \in [0, 1]$, here t is a homotopy parameter.

When K=1, we usually call it as Protruding Linear Homotopy; When K>1, when the every solution path just starts or almost ends, the lengths of step are both quite short. But the Homotopy mapping H(x,t): $R^n \times [0, 1] \rightarrow R^n$ has original equation f(x) = 0, its solution is called as original solution, then g(x) = 0 is called as target equation, and for any $t \in R$, every H（x, t）is a polynomial equation set about x .

Homotopy method may also constringe to an unexpected solution. But because of its property of explaining itself, the terminate state can be found in the process of the solving, the user can easily find out where is wrong at his initial hypothesize, in this way, they then can backtrack to former state and correct the mistake.

2.2 Homotopy iteration method

Common continuum method's calculation efficiency is quite low when it is applied in solving big under-constraint equations. Although efficient method can preclude many emanative paths, it leaves large numbers of emanative paths at most conditions; furthermore, its preprocessing course needs to specific proposal for specific problem and sometimes some skills are needed. Coefficient Homotopy method is a quite high effective method when solving the polynomial equations which has the same structure and different coefficients, but when it is applied to solving common initial equations, it still needs to adopt the common continuum method or efficient method, thus it come back to the problem of getting solutions using this two methods.

Based on calculation practice, [14],[15] brought forward a new method solving under-constraint polynomial equations, the main spirit is: (1) to construct Homotopy function $H(t, x) = (1-t) F(x) + t\gamma G(x)$ for the equations F(x)=0 which is to be solved, here G（x）is an initial function whose solutions are known. It isn't required that G（x）and F（x）are of the same efficient characters, commonly $G_i(x) = x_i^{d_i} - 1$, in which d_i is the degree of F_i（x）, t is a Homotopy parameter, γ is a random constant plural number whose imaginary part is not zero. (2) For every solution to the G（x）=0, by the Newton iteration(or any traditional iteration), calculate directly the solution of the H（t，x）=0 when t=0.5; (3) Adopt the solution as an initial value which is solved at the former step, using Newton iteration, calculate directly the solution to the H（t，x）=0 when t=0 . If the iteration emanative，then preprocess the above course from another solution to the G（x）=0 again; if the iteration is convergent, then we can get the solution to the F（x）=0. Repeat the process described above until getting all the solutions of the G（x）=0 . The common Homotopy method is 10-20 times slower than the Newton-Raphson method when it comes to non-linear equations, but in practice, the essence of following tracking precept is the application of Newton method (or other traditional iterations) many times. Traditional iteration mostly requires choosing an appropriate initial value to ensure its local convergence, however, choosing an initial value is usually of blindfold character which blocks the efficiency of the method. Now by the Homotopy function, we can resolve the problem well. The Homotopy iteration method based on the following tracking precept is appropriate for solving all polynomial equations. Due to at every step of path tracking we get rid off the estimating tache, and the number of divisor part is less, therefore the calculation efficiency is higher than the common continuum method, and the calculation complexity is also less than the common continuum method.

2.3 The Solving of Geometrical Constraint Based on Path Tracking Homotopy Iteration method

The steps of the algorithm based on the Path Tracking Homotopy Iteration method are as follow:

Step1. Assuming the constraint equations are F(x)=0, for polynomial equations F(x)=0, we can construct aided equations G（x）=0 whose full solutions are known or can be got easily, generally adopting G_i（x）$=x_i^{d_i}-1$, in which d_i is the degree of F_i（x）;

Step2. Construct Homotopy function H(t,x)=（1-t）$\bar{t}$ +tγG（x）, in which $t\in[0,1]$，$\gamma=e^{i\theta i}$ is a random constant plural number whose imaginary part is not zero;

Step3. Choose and input the iteration controlling parameter, which includes the iteration degree k, for a given precision d and emanative condition number，select the increment ⊔ t of Homotopy parameter t or the interval number n (=1/⊔ t)，and $\bar{t}$ =1-⊔ t;

Step4. Select a solution $X^{(0)}$ as an initial value to the G（x）=0;

Step5. Adopting Newton method (or another traditional numerical iteration method) , to get a solution $X^{(1)}$ of the $H(\bar{t}\ ,\ x)=0$;

Step6. If $\sum \|H(\bar{t}\ ,\ X^{(1)})\| < d$, then convert to the step9;

Step7. If $\sum \|H(\bar{t}\ ,\ X^{(1)})\| > f$, then turn to the step10;

Step8. If the times of the iteration is less than k, then let $X^{(0)} = X^{(1)}$, turn to the step5;
Step9. Set $\bar{t} = \bar{t}$ -□ t, If $\bar{t} \geq 0$, then adopt $X^{(0)} = X^{(1)}$, convert to the step5; if $\bar{t} < 0$, we can get the solution $X^{*} = X^{(1)}$ of $F(x)=0$, convert to the step11:
Step10. The process of the iteration is emanative。
Step11. If there is any solution to the G (x) =0, then convert to the step4; Else the calculation is end.

To make the above algorithm can be processed quickly and efficiently, we should select appropriate Homotopy parameter increment□ t (or the interval division number n) , the control parameter k, and the parameters d and f. Especially, the size of the interval division number n can affect directly the speed and precision of the calculation, so the choosing principle is given as follows: If the DOD of the polynomial equations is larger, then the n become larger too , in which DOD is under-degree of equations; DOD=(TD—the number of the equations) $\times$ 100%/TD。

3. RESULT

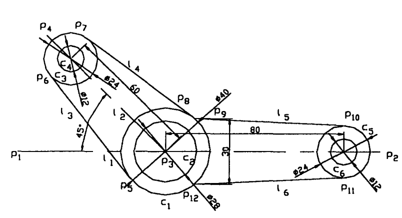

Figure 1. An example of devising

In the figure 1, there are 6 line segments and 6 circles. And 6 line segments can be decomposed 6 lines and 12 line segment nodes, so there are 24 geometric elements, the freedom degree is 54 and the constraint degree is 49. Here we change DIST_PP (p_{c1}, p_{c3}, 80) to DIST_PP (p_{c1}, p_{c3}, 60), VDIST_PP (p_9, p_{12}, 32) to VDIST_PP (p_9, p_{12}, 30), ANGLE_LL (l_1, l_2, $\pi/6$) to ANGLE_LL (l_1, l_2, $\pi/4$). If we use 49 nonlinear equations to superpose in order to solve 54 position variables corresponding to 24 geometric elements, the solving will be very inconvenient. Here we use the solving of geometrical constraint based on path tracking Homotopy iteration method, the result is as follows.

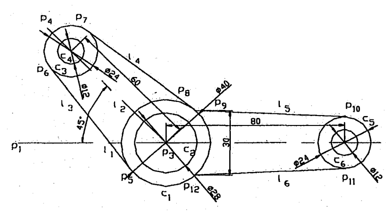

Figure 2. The solving result

We can see that when DOD=0(all the solutions of the equation needing to solve are 0), if n is larger, the area is set off more detailed, the time spent on tracking every route is more, the precision of tracking is higher, so the probability of getting all of the solutions is higher. For the under-system of DOD>0, if we set a smaller n value, the are of division is larger, we can judge many emanative routes, decrease the calculating time. For the hyper-under system (DOD $\geq$ 90%), n can be the minimum n_{min}=2, corresponding□ $t_{max}=1/n_{min}=0.5$. This is the homotopy iteration method of two-step tracking mentioned in[18],[19] . The number of emanative condition f will have an effect on the calculating efficiency and the probability of getting all the solutions will increase. The experiment indicates in most conditions the homotopy iteration method will get the same solution many times, though it can improve the reliability of getting all of the solutions, In order to decrease unnecessary iteration computation the appropriate scope of value of f is 10^3-10^{10},which should make about half of the routes converge. When there is short of experiment, we can select f $=10^7$, 10^8 firstly, then adjust by computing some routes. The select of d and k will affect computation efficiency, precision and the probability of getting all of the solutions, but they are less important than the parameters n and f.

Usually the value of d is about 10^{-6}-10^{-4}, and the value of k is small about 10-20 in the beginning of tracking(for example two-step tracking t=0.5), and the value of k is large about 200-500 before the tracking ceases(for example two-step tracking t=0).

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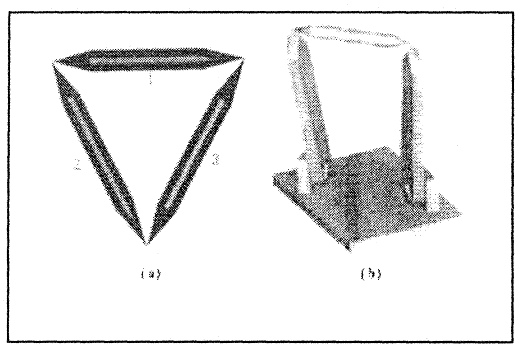

Figure 3. The example of Tracking Homotopy Iteration method

For the two examples in the figure 3, the results by different methods are indicated in table 1. The comparison of the result indicated that the solving efficiency can be improved greatly by our method.

We can see the advantages of Homotopy Iteration method are: (1) has a good universality. We can adopt a uniform solving method to the nonlinear equations; (2) can deal with the complex constraint solving; (3) when the figure has a great change, it can also achieve to convergence. (4) can have a not high demand for the iteration initialized value. The shortages are: (1) because of iteration solving, the speed is still slow. If the number of equations is large, it is not convenient to realize. (2) can't choose a solution using geometric information when there are many solutions.

Table 1. The comparison of result by different methods

	Newton Method	ordinary homotopy method	Path Tracking Homotopy Iteration method
Fig.3 (a) convergence	not convergent	convergent	convergent
time of solving(s)	-----	<100	<5
Fig.3 (b) convergence	not convergent in most conditions	convergent	convergent
time of solving(s)	<1 (if convergent)	<60	<3

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