Optimal Fuzzy Controller Mapped from LQR under Power and Torque Constraints

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Abstract. Dealing with a LQR controller surface subject to power and torque constraints, is an issue of nonlinear problem that is difficult to implement. This paper employs a fuzzy controller surface to replace the LQR surface subject to power and torque constraints by using class stacking, least square and Sugeno-type fuzzy inference mode. Through this type of transformation, called "Optimal fuzzy controller mapped from LQR", control of the system remains optimal.

Keywords: LQR, FLC, Fuzzy, Sugeno, Optimal, Class stacking.

1 Linear Quadratic Regulator (LQR)

The development of optimal control for a MIMO system [1][2] is mature nowadays. Approach of linear quadratic regulator (LQR) offers Kalman gain for state feedback to regulate a MIMO system in state-space form. But subject to power and torque limitation of motor drive, implementation of system becomes difficult using LQR approach. This section introduces some basis for LQR approach for further improvement.

1.1 LQR Approach

For a MIMO system in state-space form

$$\dot{X} = AX + Bu \tag{1}$$

where $X \in R^n$ is the state vector. The control $u \in R^w$ includes state-feedback torque in motor application, command for desired state output and disturbance that the system might encounter. Control u is thus expressed by

$$u = -KX + u_{com} + u_{dist} \tag{2}$$

where K : Kalman gain. u_{dist} : disturbance torque (colored noise). u_{com} : command for desired state output. Eq. (2) is a general form of input in LQG (Linear Quadratic

Gaussian) problems. When $\boldsymbol{u}_{dist} = \boldsymbol{o}$ and $\boldsymbol{u}_{com} \neq \boldsymbol{o}$, the system becomes a tracking problem expressed by

$$\dot{\boldsymbol{X}} = (\boldsymbol{A}\text{-}\boldsymbol{BK})\boldsymbol{X} + \boldsymbol{B}\boldsymbol{u}_{com} \tag{3}$$

At steady state it satisfies

$$\boldsymbol{O} = (\boldsymbol{A}\text{-}\boldsymbol{BK})\boldsymbol{X} + \boldsymbol{B}\boldsymbol{u}_{com} \tag{4}$$

In case $\boldsymbol{u}_{com} = \boldsymbol{o}$ in Eq. (4), the system becomes a regulator problem further called "linear quadratic regulator (LQR)" , if we select a gain $\boldsymbol{K}$, called Kalman gain such that $\boldsymbol{u} = \text{-}\boldsymbol{KX}$ to minimize the cost J in quadratic form as

$$J = \frac{1}{2}\boldsymbol{X}^{\mathrm{T}}(T)\boldsymbol{S}(T)\boldsymbol{X} + \int_0^T [\frac{1}{2}(\boldsymbol{X}^{\mathrm{T}}\boldsymbol{QX} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{Ru}) + \lambda^{\mathrm{T}}(\boldsymbol{AX} + \boldsymbol{Bu} \text{ - } \dot{\boldsymbol{X}})]dt \tag{5}$$

where $\boldsymbol{Q}$**:** dynamic state weighting, $\boldsymbol{R}$**:** control weighting $\boldsymbol{S}$: terminal state weighting and λ: Lagrange multiplier. After several algebra operations to minimize the cost, we have the following Hamiltonian matrix presenting as the transition matrix in Eq. (6):

$$\begin{bmatrix} \dot{\boldsymbol{X}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \text{-}\boldsymbol{BR}^{\text{-}1}\boldsymbol{B}^{\mathrm{T}} \\ \text{-Q} & \text{-}\boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{\lambda} \end{bmatrix} \tag{6}$$

$$\lambda(t) = \boldsymbol{S}(t)\boldsymbol{X}(t) \tag{7}$$

$$\lambda(T) = \boldsymbol{S}(T)\boldsymbol{X}(T) \tag{8}$$

$$\boldsymbol{U}(t) = \text{-}\boldsymbol{R}^{\text{-}1}\boldsymbol{B}^{\mathrm{T}}\lambda(t) = \text{-}\boldsymbol{R}^{\text{-}1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{S}(t)\boldsymbol{X}(t) = \text{-}\boldsymbol{K}(t)\boldsymbol{X}(t) \tag{9}$$

The steady gain is

$$\boldsymbol{K} = \lim_{t\to\infty}\boldsymbol{K}(t) = \boldsymbol{R}^{\text{-}1}\boldsymbol{B}^{\mathrm{T}}\lim_{t\to\infty}\boldsymbol{S}(t) = \boldsymbol{R}^{\text{-}1}\boldsymbol{B}\psi_{21}(\psi_{11})^{\text{-}1} \tag{10}$$

where $\boldsymbol{\psi}_{21}$ and $\boldsymbol{\psi}_{11}$ are the first-column block matrix of $\boldsymbol{\Psi}$ which is the modal form of the Hamiltonian matrix in Eq. (6).

1.2 Constraints of LQR Control

While implementing the optimal control $\boldsymbol{u}$ aforementioned, we always meet difficulty when subject to constraints of torque limit $\boldsymbol{u}_l$ and power limit p_l . The torque constraint, caused by stall current in motor application, is expressed by

$$\boldsymbol{u} \leq \boldsymbol{u}_l,\ \boldsymbol{u}_l = k_i\boldsymbol{i}_s \tag{11}$$

where k_i : current constant of motor. $\boldsymbol{i}_s$: stall current of motor. As usual the control $\boldsymbol{u}$ is associated with states of position and rate. Mechanical power p_m , constrained by maximum power p_l of motor, is represented by

$$p_m = \boldsymbol{u}^{\mathrm{T}}\dot{\boldsymbol{\theta}} \le p_l \tag{12}$$

where $\dot{\boldsymbol{\theta}}$ is rate-related state vector.

2 Class Stacking

From Eq. (2), the LQR control is $\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{X} = [u_1\, u_2 \cdots u_w]^{\mathrm{t}}$ where $\boldsymbol{K} = [k_{ji}] \in R^{w\times n}$ and $u_j = -\sum_{i=1}^{n} k_{ji}x_i$ for $j = 1,\cdots,w$. Since each component u_j of $\boldsymbol{u}$ will be solved one by one by the same approach of class stacking, u_j will be replaced by $u = \sum_{i=1}^{n} k_i x_i$ for illustration without loss of generality. Obviously for j^{th} component u_j of $\boldsymbol{u}$, $k_i = -k_{ji}$. The form of $u = \sum_{i=1}^{n} k_i x_i$ is a hyper plane of *n*-D. All the states x_i for $i = 1,\cdots,n$ are sorted out into two classes. One class, having *m* states, is position-related and piled up by

$$\upsilon_1 = \sum_{i=1}^{m} k_i x_i \tag{13}$$

and the other class, having *n*-*m* states, is rate-related and piled up by

$$\upsilon_2 = \sum_{i=m+1}^{n} k_i x_i \tag{14}$$

Then

$$u = \upsilon_1 + \upsilon_2 \tag{15}$$

The torque constraint is

$$\upsilon_1 + \upsilon_2 = u \le u_l \tag{16}$$

The power constraint $u\upsilon_2 \le p_l$ from Eq. (12) can be rewritten by

$$(\upsilon_1 + \upsilon_2)\upsilon_2 \le p_l \tag{17}$$

Eq. (13)~(15) thus reduce the system from *n*-D into 3-D with variables υ_1 and υ_2. Eq. (16) indicates a plane $\upsilon_1 + \upsilon_2 = u$ with known LQR control u but upper bounded by u_l. Eq. (17) indicates a nonlinear constraint. The process, called "class stacking", thus make it feasible to plot a hyper plane of *n*-D in terms of 3-D. An example using computer-aided plot is given as follows. A LQR control u, with number of total states *n*=5 and number of position-related states *m*=3, is expressed by

$$u = \sum_{i=1}^{5} k_i x_i = \upsilon_1 + \upsilon_2, \;\; \upsilon_1 = \sum_{i=1}^{3} k_i x_i \;, \upsilon_2 = \sum_{i=4}^{5} k_i x_i \qquad (18)$$

where $|x_i| \leq 1$ (normalized) and $k_i = i$ for $i = 1, \cdots, 5$. Obviously maximum of υ_1 is 6 and maximum of υ_2 is 9, subject to constraints: $\upsilon_1 + \upsilon_2 \leq 12$ (Upper bound) and $(\upsilon_1 + \upsilon_2)\upsilon_2 \leq 75$ (Inequality constraint).

By using computer-aided plot [3], Fig. 1(a)-(d) shows a sequence of transformations from LQR surface to cases of constraints are obtained. Fig. 1(a) shows the interception of a LQR surface, a torque-constrained surface and a hyperboloid due to power constraint. Fig. 1(b) shows LQR surface bounded by all constraints.

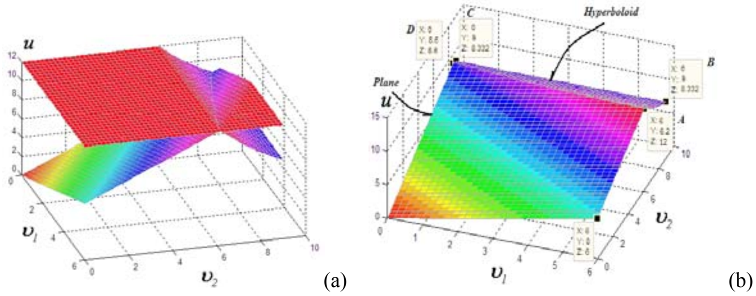

Fig. 1. Computer-aided plot (a) Interception of LQR plane, torque constraint plane and power constraint hyperboloid (b) LQR surface bounded by all constraints.

3 Fuzzy Logic Controller (FLC)

Although the approach of class stacking reduces the hyper plane from the LQR controller of *n*-D into a bounded and constrained plane in 3-D, the 3-D controller surface shown in Fig. 1.(b) is still nonlinear. Therefore we may employ a fuzzy-logic-controller (FLC) to implement this 3-D controller surface shown in Fig. 1.(b).

3.1 Fuzzy inference system (FIS)

A fuzzy logic controller (FLC) [4][5][6] is configured by processing a sequence of fuzzification, inference and defuzzification as follows. The fuzzy inference system (FIS) to be selected is Sugeno model [7] given in Fig. 2 with υ_1 and υ_2 as input fuzzy variables and the control u as known output variable which is bounded, constrained and expressed by plane *s*1: $u = \upsilon_1 + \upsilon_2$, plane *s*2: $u = 12$ (Upper bound) and hyperboloid: $u\upsilon_2 \leq 75$ (Inequality constraint).

3.2 Fuzzification

Input fuzzy variables are υ_1 and υ_2. υ_1 has five triangle membership functions, equally partitioned and spaced as shown in Fig. 2. Membership functions are defined by linguistic terms, i.e. S, SM, M, MB and B. Output variable is the control u that has five membership functions, i.e. S+, SM+, M+, MB+ and B as shown in Fig. 2.

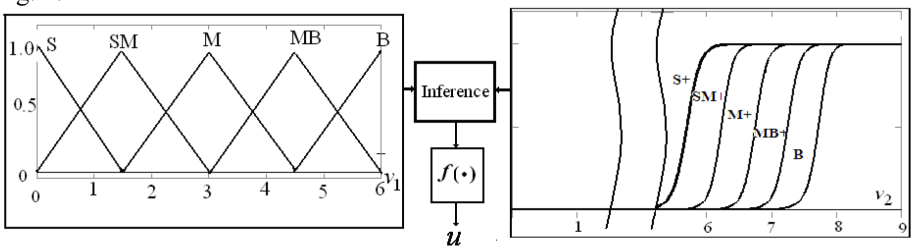

Fig. 2. Fuzzy inference system (FIS) using Sugeno model

3.3 Inference

Output variable is the control u that is composed of plane $s1$ and a hyperboloid. The plane $s1$ is $u = \upsilon_1 + \upsilon_2$. The hyperboloid $ABCD$ in Fig. 1(b) is approximately replaced by another plane in Fig. 3.(a), named $s2$ and obtained by least square with

$$s2: \quad u = \Lambda \begin{bmatrix} \upsilon_2 & \upsilon_1 & 1 \end{bmatrix}^{\mathrm{T}} \tag{19}$$

$$\Lambda = [\text{-}0.8025 \quad 0.2566 \quad 14.7253] \tag{20}$$

Λ is obtained by

$$\Lambda^{\mathrm{T}} = (\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Y})^{-1}(\boldsymbol{Y}^{\mathrm{T}}\boldsymbol{Z}) \tag{21}$$

with

$$\boldsymbol{Y} = \begin{bmatrix} \upsilon_{2A} & \upsilon_{1A} & 1 \\ \upsilon_{2B} & \upsilon_{1B} & 1 \\ \upsilon_{2C} & \upsilon_{1C} & 1 \\ \upsilon_{2D} & \upsilon_{1D} & 1 \end{bmatrix}, \boldsymbol{Z} = \begin{bmatrix} u_A \\ u_B \\ u_C \\ u_D \end{bmatrix} \tag{22}$$

The selected Λ is able to minimize error $\|\boldsymbol{Z} - \boldsymbol{Y}\Lambda\|$.

After finishing the definition of linear planes, we are able to develop the inference that includes ten rules. The former half governs the surface $s2$ and the latter half governs the surface $s1$. Rules are written as follows.

1. If (υ_1 is S) and (υ_2 is B) then (u is $s2$); 2. If (υ_1 is SM) and (υ_2 is MB+) then (u is $s2$) 3. If (υ_1 is M) and (υ_2 is M+) then (u is $s2$); 4. If (υ_1 is MB) and (υ_2 is SM+) then (u is $s2$) 5. If (υ_1 is B) and (υ_2 is S+) then (u is $s2$); 6. If (υ_1 is S) and (υ_2 is not B) then (u is $s1$) 7. If (υ_1 is SM) and (υ_2 is not MB+) then (u is $s1$); 8. If (υ_1 is M) and (υ_2 is not M+) then (u is $s1$); 9. If (υ_1 is MB) and (υ_2 is not SM+) then (u is $s1$); 10. If (υ_1 is B) and (υ_2 is not S+) then (u is $s1$).

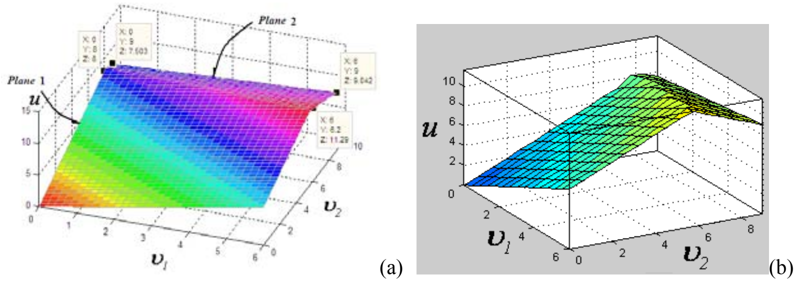

Fig. 3.(a) Hyperboloid is approximately replaced by plane *s*2 (b) Rule surface after fuzzy defuzzification.

3.4 . Defuzzification

A rule surface after defuzzification is generated as shown in Fig. 3(b) which is observed to be an approximation of Fig. 3.(a).

4 Conclusion

LQR control surface in hyperspace, subject to power and torque constraints, is difficult to implement. This paper employs class stacking to transform LQR surface in hyper space into a surface in 3-D and further applies a Sugeno-type fuzy logic controller to plot the 3-D nonlinear surface. This approach makes LQR feasible and visible.

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